Available online at http://www.journalcra.com

International Journal of Current Research
Vol. 3, Issue, 11, pp.035-039, November, 2011

INTERNATIONAL JOURNAL OF CURRENT RESEARCH

ISSN: 0975-833X

RESEARCH ARTICLE

GENERALIZED INVERSE OF K-Normal Matrix

S. Krishnamoorthy and *R. Subash

[1]Department of Mathematics, A.V.C College of Engineering, Mannampandal, Mayiladuthurai, Tamilnadu, India 609305.
[2]Department of Mathematics, Government Arts College (Autonomous), Kumbakonam, Tamilnadu, India 612 001.

ARTICLE INFO

Article History:
Received 28th August, 2011
Received in revised form
19th September, 2011
Accepted 28th October, 2011
Published online 20th November, 2011

ABSTRACT

The generalized inverses of k-normal matrix are discussed by its schur decomposition.

Key words: Schur decomposition, k-normal, k-unitary, k-diagonal, generalized inverse.

INTRODUCTION

Let $\mathbb{C}^{n\times m}$ denote the set of all complex nxm matrices. Let 'k' be a fixed product of disjoint transposition in $S_n = \{1, 2, ..., n\}$ (hence, involutary) and let 'K' be the associated permutation matrix of 'k'. Let A^* be denote the conjugate transpose matrix $A \in \mathbb{C}^{n\times m}$ and by $\mathbb{C}_r^{n\times n}$ the set of all matrices $A \in \mathbb{C}^{n\times n}$ such that $rank(A) = r$. I_n denotes the unit matrix of order n. The Moore-Penrose inverse of $A \in \mathbb{C}^{n\times m}$, is an unique matrix X satisfying the four equations

$$AXA = A\text{............}(1)$$
$$XAX = X\text{............}(2)$$
$$(AX)^* = AX\text{............}(3)$$
$$(XA)^* = XA\text{............}(4)$$

and it is denoted by $X = A^{\dagger}$. Let $A\{i, j, ..., l\}$ denote the set of matrices $X \in \mathbb{C}^{m\times n}$ which satisfy the corresponding above four equations. A matrix $X \in A\{i, j, ..., l\}$ is called an $\{i, j, ..., l\}$-inverse of A and is denoted by $A^{(i,j,...,l)}$. All of these matrices are called the generalized inverse of A. In this paper, we discuss expressions for generalized inverses of a special class of matrices, k-normal matrices, using their schur decomposition.

Definition 1.1: A matrix $A \in \mathbb{C}^{n\times n}$ is said to be k-normal, if $A\,A^* K = K\,A^* A$.

***Corresponding author:** *subash_ru@rediffmail.com*

Example 1.2: If $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ -i & 0 & 0 \end{pmatrix}$ is k-normal matrix and $X = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & -i & 0 \end{pmatrix}$. These two matrices satisfies the above four equations (1), (2), (3) and (4). Therefore X is a Moore-Penrose inverse of a singular matrix A and it is denoted by $A^{\dagger}$

Moore-Penrose inverse of k-normal matrix:

In this section {1}, {2}, {1,2}, {1,3}, {1,4}, {1,2,3}, {1,2,4}, {1,3,4}, {2,3}, {2,4}, {2,3,4}- inverses of a k-normal matrices are discussed.

Theorem 2.1: Let $A \in \mathbb{C}_r^{n\times n}$ be a k-normal matrix. Then all matrices $A^{(1)}, A^{(2)}$ are given by

(i) $$A^{(1)} = V\begin{pmatrix} \Sigma^{-1} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} V^* K$$

(ii) $$A^{(2)} = V\begin{pmatrix} \Sigma^{-1} P \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix} P^{-1} & \Sigma^{-1} P \begin{pmatrix} E \\ 0 \end{pmatrix} \\ (F \quad 0)P^{-1} & FE \end{pmatrix} V^* K,$$

where $X_{12} \in \mathbb{C}^{r\times(n-r)}$, $X_{21} \in \mathbb{C}^{(n-r)\times r}$, $X_{22} \in \mathbb{C}^{(n-r)\times(n-r)}$, $E \in \mathbb{C}^{s\times(n-r)}$ and $F \in \mathbb{C}^{(n-r)\times s}$ are arbitrary sub matrices and $0 \le s \le r$.

Proof: Let $X \in \mathbb{C}^{n\times n}$ be given by

$$KV^* XV = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \begin{matrix} r \\ n-r \end{matrix} \text{............}(5)$$

(columns: $r \quad n-r$)

(i) Using the k-unitary diagonal decomposition of A, we have that $X \in A\{1\}$ if and only if $\begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}\begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}$.

Hence $X_{11} = \Sigma^{-1}$ and X_{12}, X_{21}, X_{22} are arbitrary matrices of suitable size.

(ii) Similarly, X satisfying XAX=X if and only if

$$\begin{pmatrix} X_{11}\Sigma X_{11} & X_{11}\Sigma X_{12} \\ X_{21}\Sigma X_{11} & X_{21}\Sigma X_{12} \end{pmatrix} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$$

$$X_{11}\Sigma X_{11} = X_{11} \text{............}(5)$$

$$X_{11}\Sigma X_{12} = X_{12} \text{............}(6)$$

$$X_{21}\Sigma X_{11} = X_{21} \text{............}(7)$$

$$X_{21}\Sigma X_{12} = X_{22} \text{............}(8)$$

Pre multiplying both sides Σ in equation (5), we get $\Sigma X_{11}\Sigma X_{11} = \Sigma X_{11}$

$$\Rightarrow \quad (\Sigma X_{11})^2 = \Sigma X_{11} \text{............}(9)$$

A matrix $\Sigma X_{11} \in \mathbb{C}_s^{r\times r}$, satisfies (9) if and only if then their exist nonsingular matrix $P \in \mathbb{C}^{r\times r}$ such that $\Sigma X_{11} = P\begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix}P^{-1}$, where $0 \le s = rank(X_{11}) \le r$.

Hence $X_{11} = \Sigma^{-1}P\begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix}P^{-1}$.

Now, equations (6) and (7) have the form,

(6) $\Rightarrow$ $X_{11}\Sigma X_{12} = X_{12}$ $\Rightarrow$ $\Sigma^{-1}P\begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix}P^{-1}\Sigma X_{12} = X_{12}$ $\Rightarrow$ $P\begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix}P^{-1}\Sigma X_{12} = \Sigma X_{12}$

$\Rightarrow$ $\begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix}P^{-1}\Sigma X_{12} = P^{-1}\Sigma X_{12}$

And (7) $\Rightarrow X_{21}\Sigma X_{11} = X_{21}$ $\Rightarrow$ $X_{21}P\begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix}P^{-1} = X_{21}$

$\Rightarrow$ $X_{21}P = X_{21}P\begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix}$ from which we conclude that

$$P^{-1}\Sigma X_{12} = \begin{pmatrix} E \\ 0 \end{pmatrix}\begin{matrix} s \\ r-s \end{matrix} \text{ and}$$

$$\begin{matrix} & s & r-s \end{matrix}$$
$$X_{21}P = (F \quad 0)$$

$\Rightarrow$ $X_{12} = \Sigma^{-1}P\begin{pmatrix} E \\ 0 \end{pmatrix}$ and $X_{21} = (F \quad 0)P^{-1}$, where E, F are arbitrary sub matrices of suitable size. Substituting (8), we have $X_{22} = X_{21}\Sigma X_{12} = (F \quad 0)P^{-1}\Sigma\Sigma^{-1}P\begin{pmatrix} E \\ 0 \end{pmatrix}$

$\Rightarrow X_{22} = (F \quad 0)P^{-1}P\begin{pmatrix} E \\ 0 \end{pmatrix}$ $\Rightarrow X_{22} = (F \quad 0)\begin{pmatrix} E \\ 0 \end{pmatrix}$ $\Rightarrow$ $X_{22} = FE$.

Corollary 2.2: Let $A \in \mathbb{C}_r^{n\times n}$ be a k-normal matrix. Then any {1, 2}-inverse is given by $A^{(1,2)} = V\begin{pmatrix} \Sigma^{-1} & \Sigma^{-1}PE \\ FP^{-1} & FE \end{pmatrix}V^*K$, where $P \in \mathbb{C}_r^{r\times r}, E \in \mathbb{C}^{r\times(n-r)}$ and $F \in \mathbb{C}^{(n-r)\times r}$ are arbitrary matrices.

Proof: Considering the expressions for $A^{(1)}$ and $A^{(2)}$. From **Theorem 2.1**, we get that $\Sigma^{-1}P\begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix}P^{-1} = \Sigma^{-1}$ holds if and only if s=r, which implies that $X_{12} = \Sigma^{-1}PE$, $X_{21} = FP^{-1}$ and $X_{22} = FP^{-1}\Sigma\Sigma^{-1}PE = FE$.

Lemma 2.3: Let $A \in \mathbb{C}_r^{n\times n}$ be a k-normal matrix.

(i) Solutions of the equation (3) are given by the following general expression

$$X = V\begin{pmatrix} X_{11} & 0 \\ X_{21} & X_{22} \end{pmatrix}V^*K, \text{ where}$$

$$X_{11} = D_1 + U_1 + \Sigma^{-1}(\Sigma U_1)^*,$$
$$D_1 = diag(x_{k(1)\,k(1)},\ldots,x_{k(r)\,k(r)}).$$

$$x_{k(i)k(i)} = \begin{cases} i\,y_{k(i)k(i)}, & \lambda_{k(i)}^2 < 0 \\ y_{k(i)k(i)}, & \lambda_{k(i)}^2 > 0 \\ y_{k(i)k(i)} - \dfrac{i\,\lambda_{k(i)}^{(2)}}{\lambda_{k(i)}^{(1)}}y_{k(i)k(i)}, & \lambda_{k(i)}^2 \notin R \end{cases} \quad \ldots\ldots \; (10),$$

$y_{k(i)k(i)} \in R,\ i = 1,2,\ldots,r,\ U_1 \in \mathbb{C}^{r\times r}$ is an arbitrary strictly upper triangle matrix and X_{21}, X_{22} are arbitrary matrices of suitable size.

(ii) Solutions of the equation (4) are given by the following general expression $X = V\begin{pmatrix} \tilde{X}_{11} & \tilde{X}_{12} \\ 0 & \tilde{X}_{22} \end{pmatrix}V^*K$, where

$$\tilde{X}_{11} = D_2 + U_2 + (U_2\Sigma)^*\Sigma^{-1},$$
$$D_2 = diag(\tilde{x}_{k(1)\,k(1)},\ldots,\tilde{x}_{k(r)\,k(r)}).$$

$$\tilde{x}_{k(i)k(i)} = \begin{cases} i\,y_{k(i)k(i)} & \lambda_{k(i)}^2 < 0 \\ y_{k(i)k(i)} & \lambda_{k(i)}^2 > 0 \\ y_{k(i)k(i)} - \dfrac{i\,\lambda_{k(i)}^{(2)}}{\lambda_{k(i)}^{(1)}}y_{k(i)k(i)} & \lambda_{k(i)}^2 \notin R \end{cases},$$

$y_{k(i)k(i)} \in R,\ i = 1,2,\ldots,r,\ U_2 \in \mathbb{C}^{r\times r}$ is an arbitrary strictly upper triangle matrix and $\tilde{X}_{21}, \tilde{X}_{22}$ are arbitrary matrices of suitable size.

Proof: If X satisfies the equation (1), then $(KV^*XV)^*\begin{pmatrix} \Sigma^* & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}(KV^*XV)$ using the decomposition of X given by (5), we get $\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}^* \begin{pmatrix} \Sigma^* & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} X_{11}^* & X_{12}^* \\ X_{21}^* & X_{22}^* \end{pmatrix} \begin{pmatrix} \Sigma^* & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} X_{11}^*\Sigma^* & 0 \\ X_{21}^*\Sigma^* & 0 \end{pmatrix} = \begin{pmatrix} \Sigma X_{11} & \Sigma X_{12} \\ 0 & 0 \end{pmatrix}$$

So, $X_{11}^*\Sigma^* = \Sigma X_{11} \Rightarrow (\Sigma X_{11})^* = \Sigma X_{11}$..........(11) and $X_{12} = 0$.

Let $X_{11} = (x_{k(i)k(j)})_{r\times r}$. Then the equation (11) is equivalent to $\overline{\lambda_{k(j)} x_{k(j)k(i)}} = \lambda_{k(i)} x_{k(i)k(j)}$, $i, j = 1, 2, ..., r.$ this holds if and only if

$$\overline{\lambda_{k(i)} x_{k(i)k(i)}} = \lambda_{k(i)} x_{k(i)k(i)},\ i = 1, 2, ..., r \text{...........(12)}$$

$$x_{k(j)k(i)} = \frac{1}{\lambda_{k(j)}} \overline{\lambda_{k(i)} x_{k(i)k(j)}},\ i < j,\ i, j = 1, 2, ..., r \text{...........(13)}$$

Let $X_{11} = D_1 + U_1 + L_1$, where D_1, U_1 and L_1 are the k-diagonal, strictly upper triangle and strictly lower triangle part of X_{11}, respectively. The equation (12) holds if and only if D_1 has the form given by (10). The equation (13) is equivalent to $L_1 = \Sigma^{-1}(\Sigma U_1)^*$.

The proof of part (ii) is analogous.

Lemma 2.4: Let $A \in \mathbb{C}_r^{n\times n}$ be a k-normal matrix. Solutions of the equation (3) and (4) are given by the following general expression $X = V\begin{pmatrix} D & 0 \\ 0 & X_{22} \end{pmatrix} V^* K,$ where $D = diag(d_{k(1)\,k(1)}, ..., d_{k(r)\,k(r)})$.

$$d_{k(i)k(i)} = \begin{cases} i\, y_{k(i)k(i)} & \lambda_{k(i)}^2 < 0 \\ y_{k(i)k(i)} & \lambda_{k(i)}^2 > 0 \\ y_{k(i)k(i)} - \dfrac{i\,\lambda_{k(i)}^{(2)}}{\lambda_{k(i)}^{(1)}} y_{k(i)k(i)} & \lambda_{k(i)}^2 \notin R \end{cases},$$

$y_{k(i)k(i)} \in R,\ i = 1, 2, ..., r,$ and $X_{22} \in \mathbb{C}^{(n-r)\times(n-r)}$ is an arbitrary matrix.

Proof: If $X \in \mathbb{C}^{n\times n}$. By **lemma (2.3)**, X satisfies the equation (3) and (4) if and only if $X_{21} = 0$, $\tilde{X}_{12} = 0$, $X_{22} = \tilde{X}_{22}$, $D_1 = D_2, U_1 = U_2, \Sigma^{-1}(\Sigma U_1)^* = (U_2\Sigma)^*\Sigma^{-1}$ …………….(14).

Now, we have $U = (\Sigma^*\Sigma)^{-1} U \Sigma\Sigma^*$, where $U = U_1 = U_2$, that is

$U = diag(|\lambda_{k(1)}|^{-2}, ..., |\lambda_{k(r)}|^{-2}) U\, diag(|\lambda_{k(1)}|^2, ..., |\lambda_{k(r)}|^2)$ ……………… (15)

This equation holds if and only if U is a k-diagonal matrix. However, U is a strictly upper triangle matrix, so a necessary and sufficient condition for (15) is U=0.Taking in (10),

$$\begin{cases} y_{k(i)k(i)} = \dfrac{-1}{\lambda_{k(i)}^{(2)}}, & \lambda_{k(i)}^2 < 0, \\ y_{k(i)k(i)} = \dfrac{1}{\lambda_{k(i)}^{(1)}}, & \lambda_{k(i)}^2 > 0, \\ y_{k(i)k(i)} = \dfrac{\lambda_{k(i)}^{(1)}}{|\lambda_{k(i)}|^2}, & \lambda_{k(i)}^2 \notin R \end{cases}$$

$y_{k(i)k(i)} \in R,\ i = 1, 2, ..., r,$

We get that $D_1 = \Sigma^{-1}$, so for $U_1 = 0$. We obtain that any such solution of the equation (3) satisfies AXA=A.

Therefore, we may now pass on to expressions for the elements of A{1, 3} and A{1, 4}.

Theorem 2.5: Let $A \in \mathbb{C}_r^{n\times n}$ be a k-normal matrix. Then the elements of A{1, 3}, A{1, 4} are given by $A^{(1,3)} = V\begin{pmatrix} \Sigma^{-1} & 0 \\ X_{21} & X_{22} \end{pmatrix} V^* K,\ A^{(1,4)} = V\begin{pmatrix} \Sigma^{-1} & \tilde{X}_{12} \\ 0 & \tilde{X}_{22} \end{pmatrix} V^* K,$ respectively, where $X_{21}, X_{22}, \tilde{X}_{12}, \tilde{X}_{22}$, are arbitrary matrices of suitable size.

Proof: The proof is analogous.

Theorem 2.6: Let $A \in \mathbb{C}_r^{n\times n}$ be a k-normal matrix. Then the general forms of the elements of A{1,2, 3}, A{1,2, 4}, A{1,3, 4} are given by $A^{(1,2,3)} = V\begin{pmatrix} \Sigma^{-1} & 0 \\ FP^{-1} & 0 \end{pmatrix} V^* K,$

$$A^{(1,2,4)} = V\begin{pmatrix} \Sigma^{-1} & \Sigma^{-1}\tilde{P}E \\ 0 & 0 \end{pmatrix} V^* K,$$

$A^{(1,3,4)} = V\begin{pmatrix} \Sigma^{-1} & 0 \\ 0 & X_{22} \end{pmatrix} V^* K,$ respectively, where $P, \tilde{P} \in \mathbb{C}_r^{r\times r}, F \in \mathbb{C}^{(n-r)\times r}, E \in \mathbb{C}^{r\times(n-r)}, X_{22} \in \mathbb{C}^{(n-r)\times(n-r)}$, are arbitrary matrices.

Proof: The proof is analogous.

Theorem 2.7: Let $A \in \mathbb{C}_r^{n\times n}$ be a k-normal matrix. Then {2, 3}, {2, 4}-inverse of A are given by $A^{(2,3)} = V\begin{pmatrix} \Sigma^{-1}M_1M_1^* & 0 \\ FM_1^* & 0 \end{pmatrix} V^* K$,

$A^{(2,4)} = V\begin{pmatrix} N_1N_1^*\Sigma^{-1} & N_1E \\ 0 & 0 \end{pmatrix} V^* K$ respectively.

Where $M_1, N_1 \in \mathbb{C}^{r\times s}$ satisfy $M_1^*M_1 = I_s, N_1^*N_1 = I_s$ and $F \in \mathbb{C}^{(n-r)\times s}, E \in \mathbb{C}^{s\times(n-r)}$ are arbitrary matrices.

Proof: Let $X \in \mathbb{C}^{n\times n}$. By **Theorem 2.1 (ii) and Lemma 2.3 (i)**, we have that $X \in A\{2,3\}$ if and only if

$$D_1 + U_1 + \Sigma^{-1}(\Sigma U_1)^* = \Sigma^{-1} P \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix} P^{-1} \text{............(16)}$$

$$\left.\begin{array}{l}\Sigma^{-1}P\begin{pmatrix}E\\0\end{pmatrix}=0\\(F\quad 0)P^{-1}=X_{21}\\FE=X_{22}\end{array}\right\}\text{..........(17)}$$

First, we will prove that there exist D_1, U_1, P such that (16) holds. If we multiply the equation (16) from the left side by Σ,

We get,

$$\Sigma D_1+\Sigma U_1+(\Sigma U_1)^*=P\begin{pmatrix}I_s&0\\0&0\end{pmatrix}P^{-1}\text{...........(18)}$$

$$\Sigma D_1=(\gamma_{k(i)k(i)})_{r\times r}=\begin{cases}-\lambda^{(2)}_{k(i)}\,y_{k(i)k(i)} & \lambda^2_{k(i)}<0,\\ \lambda^{(1)}_{k(i)}\,y_{k(i)k(i)} & \lambda^2_{k(i)}>0,\\ \dfrac{\left|\lambda_{k(i)}\right|^2}{\lambda^{(1)}_{k(i)}}y_{k(i)k(i)} & \lambda^2_{k(i)}\notin R\end{cases}$$

From the equation (18) we conclude the following

(i) ΣD_1 is real k-diagonal matrix.

(ii) $\Sigma D_1+\Sigma U_1+(\Sigma U_1)^*$ is a k-hermitian matrix.

(iii) The k-eigen value set of $P\begin{pmatrix}I_s&0\\0&0\end{pmatrix}P^{-1}$ is {1, 0}.That is, $\Sigma D_1+\Sigma U_1+(\Sigma U_1)^*$ that is, $P\begin{pmatrix}I_s&0\\0&0\end{pmatrix}P^{-1}$ must be k-hermitian positive semi-definite matrix with k-eigen values 0 and 1. Because of that, the matrix P can be replaced by the k-unitary matrix M such that $M\begin{pmatrix}I_s&0\\0&0\end{pmatrix}M^*=\Sigma D_1+\Sigma U_1+(\Sigma U_1)^*=P\begin{pmatrix}I_s&0\\0&0\end{pmatrix}P^{-1}$.

Let $M=\begin{pmatrix}\overset{s}{M_1} & \overset{r-s}{M_2}\end{pmatrix}$.

Then $M\begin{pmatrix}I_s&0\\0&0\end{pmatrix}M^*=M_1M_1^*$

Denoted by $M_1M_1^*=(m_{k(i)k(i)})_{r\times r}=\Lambda_M+L_M+L_M^*$, where $\Lambda_M=\Sigma D_1, L_M^*=\Sigma U_1$

When
$$y_{k(i)k(i)}=\begin{cases}\dfrac{-m_{k(i)k(i)}}{\lambda^{(2)}_{k(i)}}, & \lambda^2_{k(i)}<0\\ \dfrac{m_{k(i)k(i)}}{\lambda^{(1)}_{k(i)}}, & \lambda^2_{k(i)}>0\\ \dfrac{\lambda^{(1)}_{k(i)}m_{k(i)k(i)}}{\left|\lambda_{k(i)}\right|^2}, & \lambda^2_{k(i)}\notin R\end{cases}$$

So, we have found $P(M), U_1, D_1$ such that the equation (16) holds.

If we put the k-unitary matrix M in (17) instead of P, we obtain that $E=0, X_{22}=0$ and $X_{21}=FM_1^*$, where F is an arbitrary matrix of suitable size.

The proof for the {2, 4}-inverse is analogous.

Theorem 2.8: let $A\in\mathbb{C}^{n\times n}_r$ be a k-normal matrix. The every {2, 3, 4}-inverse of A is of the form $A^{(2,3,4)}=V\begin{pmatrix}\Sigma^{-1}T\begin{pmatrix}I_s&0\\0&0\end{pmatrix}T^T & 0\\0&0\end{pmatrix}V^*K$, where T is a permutation matrix, $S\in\{0,1,...,r\}$.

Proof: Let $X\in\mathbb{C}^{n\times n}$ be a {2, 3, 4}-inverse of A. then $X\in A\{2\}$ and by **Lemma (2.4)**, we get that

$$D=\Sigma^{-1}P\begin{pmatrix}I_s&0\\0&0\end{pmatrix}P^{-1}\text{...........(19)}$$

$$\left.\begin{array}{l}0=\Sigma^{-1}P\begin{pmatrix}E\\0\end{pmatrix}\\0=(F\quad 0)P^{-1}\\X_{22}=FE\end{array}\right\}\text{...........(20)}$$

from (20) it follows that $E=0, F=0$ and $X_{22}=0$.

Now, we have to prove that there exist D and a non-singular matrix P such that the equation (19) holds.

By (19), we have that $\Sigma D=\Sigma^{-1}P\begin{pmatrix}I_s&0\\0&0\end{pmatrix}P^{-1}$, so ΣD, that is, $P\begin{pmatrix}I_s&0\\0&0\end{pmatrix}P^{-1}$ is a k-diagonal matrix with k-eigen values 0 or 1 and $rank(D)=s$.

Therefore, there exists a permutation matrix T such that $\Sigma D=T\begin{pmatrix}I_s&0\\0&0\end{pmatrix}T^T$.

Denote by $\Gamma=(\gamma_{k(i)k(i)})_r=T\begin{pmatrix}I_s&0\\0&0\end{pmatrix}T^T$.

$\Sigma D=\Gamma$ holds if
$$y_{k(i)k(i)}=\begin{cases}\dfrac{-\gamma_{k(i)}}{\lambda^{(2)}_{k(i)}} & \lambda^2_{k(i)}<0,\\ \dfrac{\gamma_{k(i)}}{\lambda^{(1)}_{k(i)}} & \lambda^2_{k(i)}>0,\\ \dfrac{\lambda^{(1)}_{k(i)}\gamma_{k(i)}}{\left|\lambda_{k(i)}\right|^2} & \lambda^2_{k(i)}\notin R,\end{cases}$$

Finally, we get $D=\Sigma^{-1}T\begin{pmatrix}I_s&0\\0&0\end{pmatrix}T^T$. Hence the proof.

REFERENCE

[1] Krishnamoorthy, S., and Subash, R., "On k-normal matrices", *International J. of Math. Sci. & Engg. Appls.,* 5(II) (2011), 119-130.

[2] Zheng, B., and Bapat, R.B., “Characterization of generalized inverses by a rank equation” Applied Mathematics and Computation, 151 (2004), 53-67.

[3] Chuning, He., “General forms for Moore-Penrose inverses of real symmetric matrix under the congruence decomposition condition” , Numerical mathematics(in Chinese). 28(3) (2006), 236-242.

[4] Horn, R.A., Johnson, R., “Matrix Analysis”, Cambridge University Press, 1985.

[5] Shufang, Xu., “Theory and methods of matrix computations”, Beijing University Press, 1995.

[6] Penrose, R., “A Generalized inverse for matrices” *Proc. Cambridge Philos. Soc.,* 51(1955), 406-413.

[7] Drazin, M.P., “Pseudo Inverses in Associate Rings and Semi groups” *Amer. Math. Mon,* 65(1985), 506-514.

[8] Wang, G., Wei, Y., and Qiao, S., “Generalized Inverses” Theory and Computations, Science Press (Beijing) 2004.

[9] Moore, E.H., “On the reciprocal of the general algebraic matrix (abstract)” *Bull. Amer. Math. Soc.,* 26(1920), 394-395.

****