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RESEARCH ARTICLE

NEW METHOD TO COMPUTE COMMUTING AND NON COMMUTING EXPONENTIAL MATRIX

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ABSTRACT

The matrix exponential is a very important subclass of functions of matrices that has been studied extensively in the last 50 years. In this paper, we discuss and introduce new method to compute the matrix exponential where this matrix in $M(2,R)$.

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INTRODUCTION

It is known that in the numerical fields for the exponential function $\exp(x) = e^x$ satisfies the equation of exponential function $e^{x+y} = e^x e^y$. This equality is not true in general cases when the exponential function is defined on the matrices especially when non-commutative of matrices are used. But it is known that the equation is verified if the two exponential of matrices commutative:

$$AB = BA \quad \text{then} \quad e^{A+B} = e^A e^B = e^B e^A$$

But when the application of the converse is not always true the two matrices that do not commute can apply any of the relations:

$$e^{A+B} = e^A e^B = e^B e^A$$
$$e^{A+B} \neq e^A e^B = e^B e^A$$
$$e^{A+B} = e^A e^B \neq e^B e^A$$
$$e^{A+B} \neq e^A e^B \neq e^B e^A$$

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Several studies have tried to determine the characteristics the matrices that do not commute in the exponential of matrices. In particular, the problem has been studied for 50 years for matrices of dimension two or three see for more details (Morinaga K – Nono, 1954; Bourgeois, 2007; Nathelie Smalls, 2007; Cleve Moler and Charles Van Loan, 2003; Horn and Johnson, 1991; Nicholas J. Higham and Awad H. Al-Mohy, 2010) and also taken up recently in (Bourgeois, 2007). 2009 AMS Mathematics Subject Classification: 15A16, 15A21, 15A27, 15A39. The simplest case is that of matrices in $M(2,R)$, discussed and solved in (Bourgeois, 2007) under the more general complex algebra degree two. In this paper it proposed a simple discussion on how to characterize the matrices $M(2,R)$ for which we have:

$$e^A e^B = e^B e^A = e^{A+B} \tag{1}$$

and it shows that do not exist matrices for which we have:

$$e^A e^B = e^B e^A \neq e^{A+B}$$

Definitions

The set of real matrices 2×2, $M(2,R)$ is a vector space over R with respect to the operations of matrix addition and multiplication by a real number, an algebra is not commutative respect of those operations and the usual matrix product, and is a complete metric space from the norm:

$$\|A\| = \sup_{|x|=1}|Ax|$$

In the following we will consider matrices $M(2,R)$ such that

A matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible (non-singular) if and only if the determination does not equal zero det $A = ad - bc \neq 0$ and its inverse is given by $A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

The set of invertible matrices, denoted by $GL(2,R)$ is a non-commutative group under the operation of the product of matrices, whose neutral element is the matrix:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The trace of a matrix is the sum of its elements on the main diagonal:

$$trace\ A = trace\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d$$

The commutator of two matrices A, B is the matrix which defined by:

$$[A,B] = AB - BA$$

If $[A,B] \neq 0$ then A, B and I are linearly independent. The centre $C(2,R)$ of $M(2,R)$ is the set of matrices $X \in M(2,R)$ that commute at all Matrices $M(2,R)$:

$$C(2,R) = \{X : [A,X] = 0, \forall A \in M(2,R)\}$$

and is the subgroup of $GL(2,R)$ constituted by the scalar matrices: $X = xI$ with $x \in R$. $C(2,R)$ is a Lie group and is obviously isomorphic to R^*. The sign of $X \in C(2,R)$ is the sign of $x \in R$.

Exponential of a matrix

The exponential of a matrix is defined by:

$$e^A = \sum_{k=0}^{\infty}\frac{A^k}{k!} = I + A + \frac{A^2}{2!} + \dots\dots + \frac{A^{n-1}}{(n-1)!} + \dots\dots\dots \qquad (`2)$$

The series (2) is absolutely convergent and defines an entire function in C, so it is convergent in the metric space $M(2,R)$. Since the product of matrices in $M(2,R)$ is not commutative, the exponential function so defined does not satisfy, in general, the equation (1). However, apply the following properties which we will use in the following:

$$e^A e^{-A} = e^0 = I$$

If A is invertible, then $\det A \neq 0 \Rightarrow e^{ABA^{-1}} = Ae^B A^{-1}$

$$AB = BA \Rightarrow e^{At}e^{Bt} = e^{Bt}e^{At} = e^{(B+A)t} \quad \forall t \in C$$

For more details and examples of these properties see (Morinaga and Nono, 1954; Bourgeois, 2007; Cleve Moler and Charles Van Loan, 2003; Higham, 2008; Horn and Johnson, 1991; Horn and Johnson, 1985; Syed Muhammad Ghufran, 2009; Bellman, 1995)

Main Results for computing the exponential of a matrix

The calculation of the matrix exponential $M(n,K)$ quickly becomes very complex as n increases. However, there are procedures that allow always make such a calculation in a finite number of steps, at least in principle (Cleve Moler, Charles Van Loan, 2003; Higham, 2008; Horn and Johnson, 1985; Nicholas J.Higham and Awad H. Al-Mohy, 2010; Denssis S. Bernstein and Wasin So, 1993). In the case of matrices $M(2,R)$ the calculation of the exponential is quite simple with the following decomposition.

Lemma (1)

Each matrix $A \in M(2,R)$ can be decomposed into a sum of two matrices one of which is in the centre $C(2,R)$ and the other has null trace:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = kI + A' = k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} m & b \\ c & -m \end{pmatrix}$$

And, for two matrices $A, B \in M(2,R)$, we have:

$$[A',B'] = 0 \Leftrightarrow [A,B] = 0.$$

Proof

The decomposition is obvious, just put:

$$k = \frac{a+d}{2} = \frac{trace\ A}{2},\ m = \frac{a-d}{2}$$

Then we get

$$A' = A - \frac{trace\ A}{2}I$$

Then we have

$$[A',B'] = \left[A - \frac{trace\ A}{2}I, B - \frac{trace\ B}{2}I\right] =$$

$$=[A,B]-\left[A,\frac{trace\ B}{2}I\right]-\left[\frac{trace\ A}{2}I,B\right]+\left[\frac{trace\ A}{2}I,\frac{trace\ B}{2}I\right]=[A,B]$$

Since

$$\frac{trace\ A}{2}I,\frac{trace\ B}{2}I\in C(2,R)$$

For a traceless matrix the series that defines the exponential function is particularly easy to calculate, as shown by the following lemma.

Lemma (2)

If M is a traceless matrix, where $\theta=\sqrt{\det M}$ we have:

$$e^M=I\cos\theta+M\frac{\sin\theta}{\theta}$$

Proof

Firstly we start by noting that for a traceless of matrix:

$$M=\begin{pmatrix} m & b\\ c & -m\end{pmatrix}$$

Then,

$$M^2=\begin{pmatrix} m & b\\ c & -m\end{pmatrix}\begin{pmatrix} m & b\\ c & -m\end{pmatrix}=\begin{pmatrix} m^2+bc & o\\ 0 & bc+m^2\end{pmatrix}=-\det(M)I$$

Therefore, we know $\theta=\sqrt{\det M}$ we get

$$e^M=\sum_{k=0}^{\infty}\frac{M^k}{k!}=I+M-\frac{\theta^2 I}{2!}-\frac{\theta^3 M}{3!\,\theta}.+\frac{\theta^4 I}{4!}.+\frac{\theta^5 M}{5!\,\theta}-\frac{\theta^6 I}{6!}+........=$$

$$=\mathrm{I}(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}...........)+\frac{M}{\theta}(\frac{\theta}{1!}-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}...........)=$$

$$=I\,cos\,\theta+\frac{M}{\theta}\sin\theta$$

We can therefore calculate the exponential of a matrix of $M(2,R)$ using the following Theorem.

Theorem (1)

For each matrix $A\in M(2,R)$, we have:

$$e^A=e^{kI+A'}=e^{kI}e^{A'}=e^k\left(I\cos\alpha+A'\frac{\sin\alpha}{\alpha}\right)$$

With,

$$k=\frac{trace\ A}{2},A'=A-kI,\ \alpha=\sqrt{\det A'}$$

Proof

We Use the decomposition of Lemma (1) and noting that $[kI,A']=0$, we have:

$$e^A=e^{kI+A'}=e^{kI}e^{A'}=e^{A'}e^{kI}$$

and by Lemma (2) we obtain the desired.

Notation

Note that the result of this theorem is valid even if $\det A'\le 0$. If $\det A'=0$, then $\alpha=0$, just put $\frac{\sin 0}{0}=1$ and in this case we have: $e^A=e^k(I+A')$. If $\det A'<0$, then $\alpha=\pm i|\alpha|=\pm i\sqrt{|\det A'|}$, it takes, as in the real case, the root with + sign and use the relationships between rotation functions and hyperbolic functions to obtain:

$$\frac{\sin(i|\alpha|)}{i|\alpha|}=\frac{i\sinh|\alpha|}{i|\alpha|}=\frac{e^{|\alpha|}-e^{-|\alpha|}}{2|\alpha|},\ \cos(i|\alpha|)=\cosh|\alpha|=\frac{e^{|\alpha|}+e^{-|\alpha|}}{2}$$

It proves easily that if $\mathrm{A}\in C(2,R)$ then $\mathrm{e}^{\mathrm{A}}\in C(2,R)$:

$$A=\begin{pmatrix} a & 0\\ 0 & a\end{pmatrix}\Rightarrow e^A=\begin{pmatrix} e^a & 0\\ 0 & e^a\end{pmatrix}$$

but the implication to the contrary is not true in general, as shown by the following lemma.

Lemma 3

Given a nonzero matrix:

$$A=\begin{pmatrix} a & b\\ c & d\end{pmatrix},\ e^A\in C(2,R)$$

If and only if $\mathrm{A}\in C(2,R)$

$$,\det A'=\det A-(\frac{trace\ A}{2})^2=\mu^2\pi^2 \text{ with } \mu\in N^+$$

Proof

If $\mathrm{e}^{\mathrm{A}}\in C(2,R)$ it must be:

$$e^A=e^k(I\cos\alpha+A'\frac{\sin\alpha}{\alpha})=e^k\begin{pmatrix}\cos\alpha+\frac{m\sin\alpha}{\alpha} & \frac{b\sin\alpha}{\alpha}\\ \frac{c\sin\alpha}{\alpha} & \cos\alpha-\frac{m\sin\alpha}{\alpha}\end{pmatrix}=\begin{pmatrix} x & 0\\ 0 & x\end{pmatrix}$$

by equating the terms on the secondary diagonal we have:

$$\frac{b\sin\alpha}{\alpha}=0\Rightarrow b=0 \text{ or } \frac{\sin\alpha}{\alpha}=0$$

$$\frac{c\sin\alpha}{\alpha}=0\Rightarrow c=0 \text{ or } \frac{\sin\alpha}{\alpha}=0$$

After those subtracting terms on the main diagonal, we obtain:

$$\frac{2m\sin\alpha}{\alpha} = 0 \Rightarrow m = 0 \text{ or } \frac{\sin\alpha}{\alpha} = 0$$

Now if $b = c = m = 0$ the matrix A' is zero, and then $A \in C(2, R)$.

Otherwise we must have:

$$\frac{\sin\alpha}{\alpha} = 0$$

and there are three possible causes as follow :

$$\det A' = 0 \Rightarrow \frac{\sin\alpha}{\alpha} = 1$$

$$deA' > 0 \Rightarrow \frac{\sin\alpha}{\alpha} = 0 \Rightarrow \alpha = \mu\pi \text{ with } \mu \in \mathrm{N}^+$$

$$\det A' < 0 \Rightarrow \frac{\sin\alpha}{\alpha} = \frac{\sin i|\alpha|}{i|\alpha|} = \frac{\sinh|\alpha|}{|\alpha|} > 0 \quad \forall|\alpha| > 0$$

and being $\det A' = \alpha^2$ has this structure.

Construction of matrices that verify the equation:

We want now to define a procedure to construct two matrices verifies the equation (1). Let's start with the construction of two traceless matrices, that do not commute, and such that the square root of their determinant is a positive integer multiple of π. A matrix that satisfies their
conditions has the form:

$$A_o = \mu\pi\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \mu \in N^+$$

a matrix that does not commute with A_o and that verification of the same conditions must have the form:

$$B_o = \psi\pi\begin{pmatrix} x & -y \\ \dfrac{1+x^2}{y} & -x \end{pmatrix} \qquad \psi \in N^+ \; x, y \neq 0$$

For these two matrices we have:

$$e^{A_o} = (-1)^{\mu} I \qquad \mathrm{e}^{\mathrm{B}_o} = (-1)^{\psi} I \qquad \mathrm{e}^{\mathrm{A}_o} e^{B_o} = (-1)^{\mu+\psi} e^{I}$$

Then we have:

$$A_o + B_o = (\mu+\psi)\pi\begin{pmatrix} x & -1-y \\ \dfrac{1+(1+x^2)}{y} & -x \end{pmatrix}$$

and then:

$$\det(A_o + B_o) = \left[-x^2 + (1 + y\frac{1+y+x^2}{y})\right](\mu+\psi)^2\pi^2$$

then to verify the equation (1) must be:

$$\frac{x^2}{y} + \frac{(1+y)^2}{y} = 4\nu^2 \qquad \nu \in \mathrm{N}^+$$

that is:

$$x = \sqrt{4\nu^2 y - (1+y)^2}$$

then:

$$B_o = \psi\pi\begin{pmatrix} \sqrt{4\nu^2 y - (1+y)^2} & -y \\ 4\nu^2 - y - 2 & -\sqrt{4\nu^2 y - (1+y)^2} \end{pmatrix} \qquad \psi, \nu \in N^+$$

and in case one has:

$$\det(A_o + B_o) = 4\nu^2(\mu+\psi)^2\pi^2$$

then:

$$e^{A_o+B_o} = (-1)^{2\nu(\mu+\psi)} I = (-1)^{\mu+\psi} I = e^{A_o} e^{B_o}$$

Recall now that similar matrices have the same determinant, then a date of a nonsingular matrix p can build two similar matrices to A_o and B_o such that:

$$A' = P^{-1} A_o P \qquad B' = P^{-1} B_o P$$

that still do not commute and being:

$$[A', B'] = P^{-1}[A_o, B_o]P$$

and finally two matrices A, B :

$$A = kI + A' \qquad B = hI + B'$$

such that :

$$e^{A+B} = e^A e^B = .e^B e^A = (-1)^{\mu+\psi} e^{k+h} I$$

Conclusion

There are many ways to compute the matrix exponential we present and introduce new method to compute the matrix exponential where this matrix in $M(2, R)$ and we present some lemmas and theory that explain the procedure of this method.

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