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# RESEARCH ARTICLE

# REGULAR NUMBER OF LINE BLOCK GRAPH OF A GRAPH

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**ABSTRACT**

For any $(p, q)$ graph $G$, the vertices and blocks of a graph are called its members. The line block graph $\mathrm{Lb}(G)$ of a graph $G$ as the graph whose set of vertices is the union of the set of edges and blocks of $G$ and in which two vertices are adjacent if and only if the corresponding edges of $G$ are adjacent or the corresponding members are incident. The regular number of $\mathrm{Lb}(G)$ is the minimum number of subsets into which the edge set of $\mathrm{Lb}(G)$ should be partitioned so that the subgraph induced by each subset is regular and is denoted by $r_{Lb}(G)$. In this paper some results on $r_{Lb}(G)$ were obtained and expressed in terms of elements of $G$.

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## INTRODUCTION

In this paper, we follow the notations of (Harrary, 1969). All the graphs considered here are simple, finite, and non-trivial. As usual $p$ and $q$ denote the number of vertices and edges of a graph $G$ respectively. The maximum degree of a vertex in $G$ is denoted by $(G)$. A vertex $v$ is called a cut vertex if removing it from $G$ increases the number of components of $G$. A graph $G$ is called trivial if it has no edges. The maximum distance between any two vertices in a $G$ is called a diameter and is denoted by $\mathrm{diam}(G)$. The path and tree numbers were introduced by Stanton James and Cown in (1970). The independence number $\beta_1(G)$ is the maximum cardinality of an edge independent set in G. Let $G = (V, E)$ be a graph. A set $D' \subseteq V$ is said to be a dominating set of $G$, if every vertex in $(V - D')$ is adjacent to some vertex in $D'$. The minimum cardinality of vertices in such a set is called the domination number of $G$ and is denoted by $\gamma(G)$. The dominating graph was studied by V.R.Kulli, B.Janakiram and K.M.Niranjan in (Kulli *et al*., 2004). A dominating set is said to be total dominating set of $G$, if $\mathrm{N}(D') = V$ or equivalently, if for every $v \in V$, there exists a vertex $u \in D'$, $u \neq v$, such that $u$ is adjacent to $v$. The total domination number of $G$, denoted by $\gamma_t(G)$ is the minimum cardinality of total dominating set of $G$. A set with minimum cardinality among all the maximal independent set of $G$ is called minimum independent dominating set of $G$. The cardinality of a minimum independent dominating set is called independent domination number of the graph $G$ and it is denoted by $\mathrm{i}(G)$. On complementary graphs was studied by E. A. Nordhaus and J. W. Gaddum in (1956). The regular number of graph valued function was studied by M.H.Muddebihal, Abdul Gaffar, and Shabbir Ahmed in (2015) and also developed in (Kulli *et al*., 2001; Muddebihal *et al*., 2015; Muddebihal and Abdul Gaffar, 2015; Muddebihal and Abdul Gaffar, 2016). Domination related parameters are now well studied in graph theory. The total domination $\gamma_t(G)$ was studied by M.H.Muddebihal, Srinivasa.G, and A.R.Sedamkar in (1979). Total domination in graphs was studied by E.J.Cockayne, R.M.Dawes, and S.T.Hedetniemi in (2007). This concept was studied by M.A.Henning in (2009) and was studied, for example in (Henning and Yeo, ?; Henning *et al*., 2008; Kulli, 2014; Li and Hou, 2009; Muddebihal *et al*., 2011). A dominating set $D$ of $\mathrm{L}(G)$ is a regular total dominating set (RTDS) if the induced subgraph $< D >$ has no isolated vertices and $\deg(v) = 1$, $\forall\, v \in D$. The regular total domination in line graphs was studied by M.H.Muddebihal, U.A.Panfarosh and Anil.R.Sedamkar in (2014). Total domination and total domination subdivision numbers of graphs were studied by O. Favaron, H. Karami and S. M. Sheikholeslami in (2007). On block-cutvertex trees was studied by V.R.Kulli in (1973). On

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line graphs with crossing number was studied by V.R. Kulli, D.G.Akka and L.W. Bieneke in (1979) and was studied for example in (Kulli and Patil, 1978; Kull *et al*., 1979).

## RESULTS

The following results are obvious, hence we omit its proof.

***Theorem 1*** : For any non trivial connected graph $G$, with $p \geq 3$ vertices, Lb$(G)$ is not regular.

***Proof :*** Let we discuss the regularity of line block graph Lb$(G)$ of a graph $G$. Suppose $G = K_2$ . Then Lb$(K_2)$ $K_2$ and hence regular. Now we consider the following cases.

**Case 1.** Suppose $G$ is a tree with $p \geq 3$ vertices. Then every block of L$(G)$ is complete and each block is of different regular. Since each block of $G$ is an edge, then in Lb$(G)$ each vertex is adjacent with an end vertex. Hence Lb$(G)$ is not regular.

**Case 2.** Suppose $G$ is not a tree. Then there exists at least one block which is not an edge. Let $B$ be a block of $G$ and if $G = C_3$ , then in L$(G)$ , $<$ L$(B) > = C_3$ . Hence L$(G)$ is not regular. In Lb$(G)$ $\forall$ $v_i \in$ V[L$(G)$] atleast one end edge is incident to $v_i$ , hence Lb$(G)$ is not regular.

**Case 3.** Suppose $G$ is $r$-regular where $r = 1, 2, 3, \ldots, n$ . Then we consider the following subcases.

**Subcase 3.1.** Assume $G$ is $r$ = 1, 2-regular. Then L$(G)$ is $r$ 1 and $r$-regular. For $r = 1$ regular , Lb$(G)$ is also 1 regular, which contradicts the restriction on $p \leq 2$ vertices of $G$. Suppose for $r = 2$. Then in Lb$(G)$ each vertex of L$(G)$ is adjacent with an end vertex. Hence Lb$(G)$ is not regular.

**Subcase 3.2.** Assume $G$ is $r$-regular with $r \geq 3$. Then L$(G)$ is $r + 1$ regular. Further every vertex of L$(G)$ is adjacent with an end vertex. Hence Lb$(G)$ is not regular.

In the above all cases, Lb$(G)$ is not regular with $p \geq 3$ vertices. Now, we give the sharp value of the regular number of line block graph of a path with $p \geq 4$ vertices.

***Theorem 2*** : For any path $P_p$ , with $p \geq 4$, then $r_{Lb}(P_p) = 3$.

***Proof :*** Let $P_p$ : $e_1 = v_1v_2$ , $e_2 = v_2v_3$ , $e_3 = v_3v_4$ , . . . , $e_{p-2} = v_{p-2}v_{p-1}$ , $e_{p-1} = v_{p-1}v_p$ be a path and every edge is a block, such that $B_i = e_i$ for $1 \leq i \leq p$ 1. Now in Lb$(P_p)$, $V$[Lb$(P_p)$] = { $B_1$ , $B_2$ , $B_3$ , . . . , $B_{p-1}$ }∪{$v_1'$ , $v_2'$ , $v_3'$ , . . . , $v_{p-1}'$} be the vertex set of Lb$(P_p)$. Let $E$[Lb$(P_p)$] = {$e_1''$ , $e_2''$ , $e_3''$ , . . . , $e_{p-1}''$ } ∪ {$e_1'$ , $e_2'$ , $e_3'$ , . . . , $e_{p-1}'$} be the edge set of Lb$(P_p)$, such that $e_j'' = B_jB_{j+1}$ for $1 \leq i \leq p$ 2 and $e_k' = B_kv_k'$ for $1 \leq k \leq p$ 1. Then clearly $F_1 =$ { $B_1B_2$ , $B_3B_4$ , $B_5B_6$ , . . . , $B_{p-5}B_{p-4}$, $B_{p-3}B_{p-2}$ } $F_2 =$ { $B_2B_3$ , $B_4B_5$ , $B_6B_7$ , . . . , $B_{p-4}B_{p-3}$ , $B_{p-2}B_{p-1}$ } and $F_3 =$ { $B_1v_1'$ , $B_2v_2'$ , $B_3v_3'$ , . . . , $B_{p-2}v_{p-2}'$, $B_{p-1}\, v_{p-1}'$ }. Let $F$ be the minimum regular partition of Lb$(P_p)$. Then,

$$r_{Lb}(P_p) = |\{ F_1 , F_2 , F_3 \}|$$
$$r_{Lb}(P_p) = |F|$$
$$r_{Lb}(P_p) = 3.$$

In the next result we obtain the exact value of the regular number of line block graph of a complete graph.

***Theorem 3***: For any complete graph $K_p$ with p $\geq$ 2, then $r_{Lb}(K_p) = \frac{p(p-1)}{2}$ .

***Proof :*** Let $v_1$ , $v_2$ , $v_3$ , . . . , $v_{p-1}$ , $v_p$ be the vertices of $K_p$ and each vertex is of degree $p$ 1. Let $e_1$ , $e_2$ , $e_3$ , . . . , $e_{\frac{p(p-1)}{2}}$ be the edges of $K_p$ such that $e_1 = v_1v_2$ , $e_2 = v_1v_3$ , $e_3 = v_1v_4$ , . . . , $e_{\frac{p(p-1)}{2}} = v_{p-1}v_p$. Now, it is known that every $K_p$ itself is a single block and it represents as a single vertex in Lb[$K_p$] say $B_1$ . Now, in Lb[$K_p$], V[Lb$(K_p)$] = {$B_1$}∪{$e_1$ , $e_2$ , $e_3$ , . . . , $e_{\frac{p(p-1)}{2}}$} , i.e. $e_1$ , $e_2$ , $e_3$ , . . . , $e_{\frac{p(p-1)}{2}}$ be the vertices adjacent to $B_1$. Let $e_1'$ , $e_2'$ , $e_3'$ , . . . , $e_{\frac{p(p-1)}{2}}'$ be the edges of Lb[$K_p$], such that $e_i' = B_1e_i$ for $1 \leq i \leq \frac{p(p-1)}{2}$ . Let $F$ be the minimum regular partition of Lb[$K_p$]. Since Lb[$K_p$] is a star ,the subgraph induced by each subset of $F$ is $mK_2$ with $m \geq 1$. Therefore, each of the edges $B_1e_1$ , $B_1e_2$ , $B_1e_3$ , . . . , $B_1e_{\frac{p(p-1)}{2}}$ , belongs to different sets $F_1$ , $F_2$ , $F_3$ , . . . , $F_{\frac{p(p-1)}{2}}$ of $F$ respectively. Also, each edge not incident with $B_1$ belongs to one of $F_1$ , $F_2$ , $F_3$ , . . . , $F_{\frac{p(p-1)}{2}}$ .

Hence,

$$r_{Lb}(K_p) = |F|$$
$$= \frac{p(p-1)}{2} .$$

Next, we prove the following result to prove our next result.

***Theorem 4 :*** For any graph $G$, $r_{Lb}(G) \leq q$ $\beta_1(G) + 1$.

***Proof :*** Let $S$ be a maximum edge independent set in $G$. Then $E$ $S$ has at most $|E$ $S|$ edge independent sets.

Thus,

$$r_{Lb}(G) \leq |E \quad S| + 1.$$
$$r_{Lb}(G) \leq q \quad \beta_1(G) + 1.$$

Now, the following result determines the upper bound on $r_{Lb}(G)$.

***Theorem 5*** : For any non-trivial graph $G$, then $r_{Lb}(G) \leq 2q$ $p + 1$.

***Proof*** : By Theorem 4, we have

$$r_{Lb}(G) \leq q \quad \beta_1(G) + 1.$$

Since, $\beta_1(G) \geq \gamma'(G)$.

Where $\gamma'(G)$ is the edge domination number of $G$.
This implies,

$r_{Lb}(G) \leq q \quad \gamma'(G) + 1.$
Also, $p \quad q \leq \gamma'(G)$.

Thus,
$r_{Lb}(G) \leq q \quad (p \quad q) + 1.$
$r_{Lb}(G) \leq q \quad p + q + 1.$
$r_{Lb}(G) \leq 2q \quad p + 1.$
In the next result we obtain Nordhaus-Gaddum type result on $r_{Lb}(P_p)$.

***Theorem 6*** : For any path $P_p$, with $p \geq 4$, then $r_{Lb}(P_p) + r_{Lb}(\overline{P_p}) \leq p(p \quad 3) + 2.$

***Proof*** : By Theorem 5, we have

$r_{Lb}(P_p) \leq 2q \quad p + 1.$
$r_{Lb}(\overline{P_p}) \leq 2\bar{q} \quad p + 1.$
$r_{Lb}(P_p) + r_{Lb}(\overline{P_p}) \leq 2(q + \bar{q}) \quad 2p + 2.$
$r_{Lb}(P_p) + r_{Lb}(\overline{P_p}) \leq 2\binom{p}{2} \quad 2p + 2.$
$r_{Lb}(P_p) + r_{Lb}(\overline{P_p}) \leq 2\frac{p(p-1)}{2} \quad 2p + 2.$
$r_{Lb}(P_p) + r_{Lb}(\overline{P_p}) \leq p(p \quad 1) \quad 2p + 2.$
$r_{Lb}(P_p) + r_{Lb}(\overline{P_p}) \leq p(p \quad 3) + 2.$

In the following theorem we determine the exact value of a regular number of line block graph of a star.

***Theorem 7*** : For any star $K_{1,n}$, with $n \geq 2$, then $r_{Lb}(K_{1,n}) = 2$.

***Proof*** : Let $v_1, v_2, v_3, \ldots, v_n, v_{n+1}$ be the vertices of $K_{1,n}$ such that $\deg(v_{n+1}) = n$, and $\deg(v_1) = \deg(v_2) = \deg(v_3) = , \ldots, \deg(v_n) = 1$. In $K_{1,n}$ every edge is a block such that $B_1 = e_1 = v_1v_{n+1}$, $B_2 = e_2 = v_2v_{n+1}$, $B_3 = e_3 = v_3v_{n+1}, \ldots, B_n = e_n = v_nv_{n+1}$, and each edge is adjacent to each other. Then clearly the blocks $B_1, B_2, B_3, \ldots, B_n$ becomes the vertices in $\mathrm{Lb}(K_{1,n})$. Let $v_1', v_2', v_3', \ldots, v_n'$ be the end vertices of $\mathrm{Lb}(K_{1,n})$ which are adjacent to $B_1, B_2, B_3, \ldots, B_n$ respectively. Then clearly in $\mathrm{Lb}(K_{1,n})$, $V[\mathrm{Lb}(K_{1,n})] = \{B_1, B_2, B_3, \ldots, B_n\} \cup \{v_1', v_2', v_3', \ldots, v_n'\}$. The vertices $B_1, B_2, B_3, \ldots, B_n$ forms a complete graph with $(n \quad 1)$ regular. Let $F_1 = \{B_1, B_2, B_3, \ldots, B_n\}$ is a single partition of a complete graph and $F_2 = \{B_1v_1', v_2', B_3v_3', \ldots, B_nv_n'\}$. Let $F$ be the minimum regular partition.

Hence,

$r_{Lb}(K_{1,n}) = |F|$
$r_{Lb}(K_{1,n}) = 2.$

Next, we develop the regular number of line block graph for a wheel.

***Theorem 8*** : For any wheel $W_p$, with $p \geq 4$ vertices, then $r_{Lb}(W_p) = 2p \quad 2$.

***Proof :*** Let $v_1, v_2, v_3, \ldots, v_p$ be the vertices of $W_p$ such that $\deg(v_i) = 3$ for $1 \leq i \leq p \quad 1$ and $\deg(v_p) = p \quad 1$. Let $\{e_1 = v_1', e_2 = v_2', e_3 = v_3', \ldots, e_{p-1} = v_{p-1}'\}$, $\{e_1' = v_1'', e_2' = v_2'', e_3' = v_3'', \ldots, e_{p-1}' = v_{p-1}''\}$ be the edges of $W_p$ such that $e_i = v_iv_{i+1}$ for $1 \leq i \leq p \quad 2$, $e_{p-1} = v_1v_{p-1}$ and $e_i' = v_iv_p$ for $1 \leq i \leq p \quad 1$. Since every $W_p$ itself is a single block and in $\mathrm{Lb}(W_p)$ it represents as a single vertex, say $B_1$. Then clearly, in $\mathrm{Lb}(W_p)$, we have $V[\mathrm{Lb}(W_p)] = \{B_1\} \cup \{v_1', v_2', v_3', \ldots, v_{p-1}'\} \cup \{v_1'', v_2'', v_3'', \ldots, v_{p-1}''\}$ and the vertices $v_1', v_2', v_3', \ldots, v_{p-1}'$ and $v_1'', v_2'', v_3'', \ldots, v_{p-1}''$ are adjacent to $B_1$. Let $e_1'', e_2'', e_3'', \ldots, e_{p-1}''$ and $e_1''', e_2''', e_3''', \ldots, e_{p-1}'''$ be the edges of $\mathrm{Lb}(W_p)$, such that $e_j'' = B_1e_j$ for $1 \leq j \leq p \quad 1$ and $e_k''' = B_1e_k'$ for $1 \leq k \leq p \quad 1$. Let $F$ be the minimum regular partition of $\mathrm{Lb}(W_p)$. Since, $\mathrm{Lb}(W_p)$ is a star, the subgraph induced by each subset of $F$ is $mK_2$ with $m \geq 1$. Therefore each of the edges $B_1e_1$, $B_1e_2$, $B_1e_3, \ldots, B_1e_{p-2}$, $B_1e_{p-1}$, $B_1e_1'$, $B_1e_2', \ldots, B_1e_{p-2}'$, $B_1e_{p-1}'$ belongs to different sets $F_1, F_2, F_3, \ldots, F_{2p-2}$ respectively. Also each edge not incident with $B_1$ belongs to one of $F_1, F_2, F_3, \ldots, F_{2p-2}$.

Hence,

$r_{Lb}(W_p) = |F|.$
$r_{Lb}(W_p) = 2p \quad 2.$

Now, we establish the sharp value for $r_{Lb}(G)$ of a cubic graph.

***Theorem 9*** : For any $(p, q)$ cubic graph $G$, with $p \geq 4$, then $r_{Lb}(G) = \frac{3p}{2}$.

***Proof*** : Let $v_1, v_2, v_3, \ldots, v_{p-1}, v_p$ be the vertices of a cubic graph such that $\deg(v_i) = 3$ for $1 \leq i \leq p$. Let $e_1, e_2, e_3, \ldots, e_{\left(\frac{3p}{2}-1\right)}, e_{\left(\frac{3p}{2}\right)}$ be the edges of a cubic graph where $e_1 = v_1v_2 = v_1'$, $e_2 = v_2v_3 = v_2'$, $e_3 = v_3v_4 = v_3'$, $\ldots, e_{\left(\frac{p}{2}-1\right)} = v_{\left(\frac{p}{2}-1\right)}v_{\left(\frac{p}{2}\right)} = v_{\left(\frac{p}{2}-1\right)}'$, $e_{\left(\frac{p}{2}\right)} = v_{\left(\frac{p}{2}\right)}v_1 = v_{\left(\frac{p}{2}\right)}'$, further $e_{\left(\frac{p}{2}+1\right)} = v_{\left(\frac{p}{2}+1\right)}v_{\left(\frac{p}{2}+2\right)} = v_{\left(\frac{p}{2}+1\right)}'$, $e_{\left(\frac{p}{2}+2\right)} = v_{\left(\frac{p}{2}+2\right)}v_{\left(\frac{p}{2}+3\right)} = v_{\left(\frac{p}{2}+2\right)}'$, $e_{\left(\frac{p}{2}+3\right)} = v_{\left(\frac{p}{2}+3\right)}v_{\left(\frac{p}{2}+4\right)} = v_{\left(\frac{p}{2}+3\right)}'$, $\ldots, e_{p-1} = v_{p-1}v_p = v_{p-1}'$, $e_p = v_pv_{\left(\frac{p}{2}+1\right)} = v_p'$ and $e_{(p+1)} = v_1v_{\left(\frac{p}{2}+1\right)} = v_{p+1}'$, $e_{(p+2)} = v_2v_{\left(\frac{p}{2}+2\right)} = v_{p+2}'$, $e_{(p+3)} = v_3v_{\left(\frac{p}{2}+3\right)} = v_{p+3}'$, $\ldots, e_{\left(\frac{3p}{2}-1\right)} = v_{\left(\frac{p}{2}-1\right)}v_{p-1} = v_{\left(\frac{3p}{2}-1\right)}'$, $e_{\left(\frac{3p}{2}\right)} = v_{\left(\frac{p}{2}\right)}v_p = v_{\left(\frac{3p}{2}\right)}'$. Since, every cubic graph $G$ itself is a single block. Now, in $\mathrm{Lb}(G)$, let the block cubic graph is represented as the vertex $B_1$, and $e_1, e_2, e_3, \ldots, e_{\left(\frac{3p}{2}\right)}$ be the corresponding vertices of the edges which are in $G$. Then, clearly in $\mathrm{Lb}(G)$, $V[\mathrm{Lb}(G)] = \{B_1\} \cup \{e_1, e_2, e_3, \ldots, e_{\left(\frac{p}{2}-1\right)}, e_{\left(\frac{p}{2}\right)}, e_{\left(\frac{p}{2}+1\right)}, e_{\left(\frac{p}{2}+2\right)}, \ldots, e_{p-1}, e_p, e_{(p+1)}, e_{(p+2)}, \ldots, e_{\left(\frac{3p}{2}-1\right)}, e_{\left(\frac{3p}{2}\right)}\}$ which are adjacent to $B_1$. Let $e_1'$, $e_2'$, $e_3'$, $\ldots, e_{\left(\frac{p}{2}-1\right)}'$, $e_{\left(\frac{p}{2}\right)}'$, $e_{\left(\frac{p}{2}+1\right)}'$, $e_{\left(\frac{p}{2}+2\right)}'$, $\ldots, e_{p-1}'$, $e_p'$, $e_{p+1}'$, $e_{p+2}'$, $\ldots, e_{\left(\frac{3p}{2}-1\right)}'$, $e_{\left(\frac{3p}{2}\right)}'$ be the edges of

Lb($G$), such that $e_i' = B_1 e_i$ for $1 \le i \le \frac{p}{2}$ , $e_j' = B_1 e_j$ for $\frac{p}{2} + 1 \le j \le p$ , $e_k' = B_1 e_k$ for $p + 1 \le k \le \frac{3p}{2}$ . Let $F$ be the minimum regular partition of Lb($G$). Since, Lb($G$) is a star, the subgraph induced by each subset of $F$ is $mK_2$ with $m \ge 1$. Therefore each of the edges $B_1e_1$ , $B_1e_2$ , $B_1e_3$ , . . . , $B_1e_{\left(\frac{p}{2}-1\right)}$ , $B_1e_{\left(\frac{p}{2}\right)}$ , $B_1e_{\left(\frac{p}{2}+1\right)}$ , $B_1e_{\left(\frac{p}{2}+2\right)}$ , . . . , $B_1e_{p-1}$ , $B_1e_p$ , $B_1e_{(p+1)}$ , $B_1e_{(p+2)}$ , . . . , $B_1e_{\left(\frac{3p}{2}-1\right)}$ , $B_1e_{\left(\frac{3p}{2}\right)}$ belongs to different sets $F_1$ , $F_2$ , $F_3$ , . . . , $F_{\left(\frac{3p}{2}\right)}$ respectively. Also each edge not incident with $B_1$ belongs to one of $F_1$ , $F_2$ , $F_3$ , . . . , $F_{\left(\frac{3p}{2}\right)}$ .

Hence,

$r_{Lb}(G) = |F|$.
$r_{Lb}(W_p) = \frac{3p}{2}$.

Next, we obtain the exact value for regular number of line block graph of a complete bipartite graph.

***Theorem 10***: For any non-trivial complete bipartite graph $K_{m,n}$ with $m \ne 1$ *or* $n \ne 1$, then

$r_{Lb}(K_{m,n}) = mn$.

***Proof*** : Let $v_1$ , $v_2$ , $v_3$ , . . . , $v_{m-1}$ , $v_m$ , $v_{m+1}$ , $v_{m+2}$ , . . . , $v_{m+n-1}$ , $v_{m+n}$ be the vertices of $K_{m,n}$ such that $\deg(v_i) = n$ , for $1 \le i \le m$ and $\deg(v_j) = m$ , for $m + 1 \le j \le m + n$ . Let $e_1 = v_1 v_{m+1}$, $e_2 = v_1 v_{m+2}$ , . . . , $e_{mn-1} = v_m v_{m+n-1}$ , $e_{mn} = v_m v_{m+n}$ be the edges of $K_{m,n}$ . Since every $K_{m,n}$ itself is a single block in Lb($K_{m,n}$) and say $B_1$ . Now, in Lb($K_{m,n}$) , $V$[Lb($K_{m,n}$)] = $\{B_1\} \cup \{e_1, e_2, e_3, \ldots, e_{mn-1}, e_{mn}\}$ , and the vertices $e_1, e_2, e_3, \ldots, e_{mn-1}, e_{mn}$ are adjacent to $B_1$ . Let $e_1'$ , $e_2'$ , $e_3'$ , . . . , $e_{mn-1}'$ , $e_{mn}'$ be the edges of Lb($K_{m,n}$). Let $F$ be the minimum regular partition of Lb($K_{m,n}$). Since Lb($K_{m,n}$) is a star, the subgraph induced by each subset of $F$ is $mK_2$ with $m \ge 1$. Therefore each of the edges $B_1e_1$ , $B_1e_2$ , $B_1e_3$ , . . . , $B_1e_{mn-1}$ , $B_1e_{mn}$ belongs to different sets $F_1$ , $F_2$ , $F_3$ , . . . , $F_{mn}$ respectively. Also each edge not incident with $B_1$ belongs to one of $F_1$ , $F_2$ , $F_3$ , . . . , $F_{mn}$ .

Hence,

$r_{Lb}(K_{m,n}) = |F|$.
$r_{Lb}(K_{m,n}) = mn$.

Now, we develop the result which establish the relationship between $r_{Lb}(G)$ and diam ($G$).

***Theorem 11***: For any non-trivial graph $G$, then $r_{Lb}(G) \le q$ diam($G$) + 3.

***Proof :*** Let $P_n$ : $v_1v_2$ , $v_2v_3$ , $v_3v_4$ , . . . , $v_{n-1}v_n$ be a diametral path with diam($G$) + 1 vertices.Since each edge is a block in $G$ such that $B_i = v_i v_{i+1}$ for $1 \le i \le n$ 1 . Now in Lb($P_n$), $V$[Lb($P_n$)] = $\{B_1, B_2, B_3, \ldots, B_{n-1}\} \cup \{v_1', v_2', v_3', \ldots, v_{n-1}'\}$. Then, clearly $F_1 = \{B_1B_2, B_3B_4, B_5B_6, \ldots, B_{n-3}B_{n-2}\}$ $F_2 = \{B_2B_3, B_4B_5, B_6B_7, \ldots, B_{n-2}B_{n-1}\}$ and $F_3 = \{B_1v_1', B_2v_2', B_3v_3', \ldots, B_{n-1}v_{n-1}'\}$ Let $F$ be the minimum regular partition of Lb($P_n$).

Thus, $r_{Lb}(G) \le |F|$
$r_{Lb}(G) \le q$ ($n$ 1) + 3.
$r_{Lb}(G) \le q$ diam($G$) + 3.

In the following theorem determine the exact value for the regular number of line block graph of a tree $T$, where $T$ is a non-trivial tree with $n$-cutvertices with same degree.

***Theorem 12***: For any non-trivial tree $T$, with $n$ cutvertices with same degree, then $r_{Lb}(T) = 3$.

***Proof :*** Let $V_1 = \{v_1, v_2, v_3, \ldots, v_m\}$ be the set of cut vertices with same degree in $T$. Let $V_2 = \{v_{m+1}, v_{m+2}, v_{m+3}, \ldots, v_{m+n}\}$ be the set of end vertices of $T$. Then clearly $V(T) = V_1 \cup V_2$ . Since in any tree every edge is a block. Let $V_1 = \{v_1, v_2, v_3, \ldots, v_m\}$ be the set of cut vertices with same degree in $T$ such that $\deg(v_1) = \deg(v_2) = \deg(v_3) =$, . . . , $= \deg(v_m) =$ say $r$. Then clearly in Lb($T$) has $m$ blocks and each block is ($r$ 1) regular. Since L($T$) ⊂ Lb($T$), then each block of L($T$) is complete and every cut vertex of L($T$) lies on exactly two blocks. Let $\{v_1', v_2', v_3', \ldots, v_n'\}$ be the set of cut vertices in L($T$) and $\{b_1, b_2, b_3, \ldots, b_k\}$ be the set of blocks in L($T$) which are complete. In L($T$), N($v_i'$) = $\{b_i\} \cup \{b_j\}$ $\forall$ $i = j = 1$. Hence these blocks can be partitioned in two sets say $F_1$ and $F_2$ , and the remaining edges $mK_2$ which are incident on the vertices of these blocks are belongs to $F_3$ . Let $F$ be the minimum regular partition of Lb($T$).

Thus,

$r_{Lb}(T) = |F|$.
$r_{Lb}(T) = 3$.

Now, we develop the exact value of $r_{Lb}(T)$ with $m$-distinct cutvertices of a non-trivial tree $T$.

***Theorem 13***: For any non-trivial tree $T$, with $m$-distinct cutvertices, then $r_{Lb}(T) = m + 1$.

***Proof :*** Let $V_1 = \{v_1, v_2, v_3, \ldots, v_m\}$ be the set of distinct cut vertices in $T$. Let $V_2 = \{v_{m+1}, v_{m+2}, v_{m+3}, \ldots, v_{m+n}\}$ be the set of end vertices in $T$. Then clearly $V(T) = V_1 \cup V_2$.Since every edge is a block in $T$. Suppose $\deg(v_1) < \deg(v_2) < \deg(v_3) < \ldots < \deg(v_m)$ which are the cut vertices in $T$. Then in Lb($T$) has $m$-blocks and each block is a complete graph with distinct degrees. Therefore, each block belongs to different sets $F_1$ , $F_2$ , $F_3$ , . . . , $F_m$ . Further the remaining edges $mK_2$ can be placed in a single partition.

Hence,

$r_{Lb}(T) = |\{F_1, F_2, F_3, \ldots, F_m\}| + 1$.
$r_{Lb}(T) = m + 1$.
Next, we establish the relationship between $r_{Lb}(T)$ and $\gamma(T)$, where $T$ is a non-trivial tree with $n$-cutvertices with same degree.

***Theorem 14***: For any non-trivial tree $T$, with $n$-cutvertices with same degree, and $n \geq 3$, then

$$r_{Lb}(T) \leq \gamma(T).$$

***Proof*** : For any tree $T$, let $D = \{ v_1, v_2, v_3, \ldots, v_n \}$ be the set of non-end vertices in $T$. Suppose every vertex of $D$ is adjacent to atleast one end vertex. Then $D$ itself is a $\gamma$-set of $T$. Suppose $\deg(v_1) = \deg(v_2) = \deg(v_3) = \ldots = \deg(v_n) = r$ (say). Assume $\forall\ v_i \in D$ such that $1 \leq i \leq 3$ are adjacent. Then in Lb($T$), $\forall\ v_i \in D$ gives 3 blocks and each block is $(r \quad 1)$ regular and adjacent to each other.

By Theorem 12, we have

$$r_{Lb}(T) = 3. \text{ Clearly, } r_{Lb}(T) = \gamma(T) = 3.$$

For inequality, suppose $\gamma(T) > 3$ , then the dominating set $D_1 = \{ v_1, v_2, v_3, \ldots, v_j \}$. Let $D_1 \subset D$ and $\forall\ v_j \in D_1$ such that $1 \leq j \leq n$ are adjacent. Then in Lb(T), $\forall\ v_j \in D$ gives $n$ number of blocks and each block is $(r \quad 1)$ regular and adjacent to each other.

Thus,
By Theorem 12, we have,
$r_{Lb}(T) = 3$, but $\gamma(T) > 3$.

Hence,
$r_{Lb}(T) \leq \gamma(T)$.

Now, we obtain the relationship between $r_{Lb}(T)$ and $\gamma(T)$. Where $T$ is a non-trivial tree with $n$-distinct cutvertices.

***Theorem 15***: For any non-trivial tree $T$, with $n$-distinct cutvertices, then $r_{Lb}(T) = \gamma(T) + 1$.

***Proof*** : For any tree $T$, let $D = \{ v_1, v_2, v_3, \ldots, v_n \}$ be the set of non-end vertices which are cut vertices in $T$. Suppose every vertex of $D$ is adjacent to atleast one end vertex. Then $D$ is a $\gamma$-set of $T$. Suppose $\deg(v_1) > \deg(v_2) > \deg(v_3) > \ldots > \deg(v_n)$. Then in Lb($T$), $\forall\ v_i \in D$ such that $1 \leq i \leq n$ gives $n$-blocks and each block is a complete graph with distinct degrees. Hence, each block belongs to different sets $F_1$, $F_2$, $F_3, \ldots, F_n$ . And each end edge of the Lb($T$) is $K_2$ and incident on the vertices of the blocks. And these edges can be partitioned in a single set.

Hence,

$r_{Lb}(T) = |\{ F_1, F_2, F_3, \ldots, F_n \}| + 1.$
$r_{Lb}(T) = \gamma(T) + 1.$

In the following theorem we establish the relationship between $r_{Lb}(T)$ and $\gamma_t(T)$.

***Theorem 16***: For any non-trivial tree $T$, with $n$-cutvertices with same degree, $n \geq 3$ and $p \geq 5$, then

$r_{Lb}(T) \leq \gamma_t(T)$.

***Proof*** : For a non-trivial tree $T$ with $p = 2, 3$ , $\gamma_t(T) = 2$ , where as for $p = 2$, $r_{Lb} = 1$ and if $p = 3$, $r_{Lb}(T) = 2$. Hence we consider $p \geq 3$. Suppose S = $\{ v_1, v_2, v_3, \ldots, v_k \} \subseteq V(T)$ be the minimal set of vertices which covers all the vertices of $T$. If the subgraph $< S >$ has no isolated vertices, then $S$ forms a $\gamma_t$-set of $T$. Otherwise, there exists at least one vertex $v \in$ N($S$) such that $S \cup \{v\}$ forms a minimal total dominating set of $T$. Let H = $\{ v_1, v_2, v_3, \ldots, v_n \}$ be the set of cut vertices of $T$ such that $\deg(v_1) = \deg(v_2) = \deg(v_3) = \ldots = \deg(v_n) = r$. Now without loss of generality, since $V$[L($T$)] = E($T$), let $F = \{ v_1, v_2, v_3, \ldots, v_n \} = V$[L($T$)] be the set of vertices corresponding to the edges incident to the vertices of H in $T$. Further the edges which are incident with the vertices of H in $T$ forms a complete subgraph with $r \quad 1$ regular. Let $F' \subseteq F$ be the set of cut vertices in L($T$) and $\forall\ v_i \in F'$ lies on exactly two blocks of L($T$). Let M = $\{ B_1, B_2, \ldots, B_n \}$ be the set of blocks in L($T$) such that N($v_i$) $= B_i \cup B_j$ , each $B_i \in F_1$ and $B_j \in F_2$ , where $F_1$ and $F_2$ are the regular partition of $E$[L($T$)]. Since each edge is a block , then each $v_i \in F$ is incident to an edge in Lb($T$). By the definition of Lb($G$), let $\{ e_1, e_2, e_3, \ldots, e_q \}$ be the set of edges which corresponds to the blocks of $T$. Now in Lb($T$), $V$[Lb(T)] = $V$[L($T$)]$\cup \{ e_1, e_2, e_3, \ldots, e_q \}$ , such that each $e_i$ $1 \leq i \leq k$ is adjacent each $v_i \in F$, forms a set L = $\{ e_1', e_2', e_3', \ldots, e_q' \}$ which are edge-disjoint end edges in Lb($T$). Hence the set L belongs to a regular partition $F_3$ . Let $F$ be the minimum regular partition of Lb($T$). Let $D = \{ v_1, v_2, v_3, \ldots, v_n \} \subseteq V(T)$ be the minimal dominating set of $T$. Suppose $< D >$ has no isolates. Then $D$ is a $\gamma_t$-set. Otherwise there exists a vertex $u \in V(T) \quad D$ such that $D \cup \{u\}$ forms dominating set. Now $< D \cup \{u\} >$ has no isolates. It follows that $r_{Lb}(T) \leq |D|$.

Hence,

$r_{Lb}(T) \leq \gamma_t(T)$.

In the next result, we balanced $r_{Lb}(T)$ and $r_L(T)$.

***Theorem 17***: For any non-trivial tree $T$, with $n$-cutvertices with same degree, then

$r_{Lb}(T) = r_L(T) + 1$.

***Proof*** : Let $G$ be a non-trivial tree. Then L($T$) $\subset$ Lb($T$). In L($T$) each cutvertex lies on exactly two blocks and belongs to $F_1$ and $F_2$. Let H = $\{ e_1, e_2, e_3, \ldots, e_j \}$ be the set of end edges in Lb($T$). Since each edge is a block in $G$. Then exactly one $e_i$, $1 \leq i \leq j$ is incident with exactly one vertex of block of L($T$), such that $e_1 \cap e_2 \cap e_3 \cap \ldots \cap e_j =$ . Hence H $\in F_3$ . Now, $r_L(T) = |\{ F_1, F_2\}|$ and

$r_{Lb}(T) = |\{ F_1, F_2\} \cup F_3|$ which gives,

$$r_{Lb}(T) = r_L(T) + 1.$$

Finally, in our last result, we discuss the relationship between $r_{Lb}(T)$ and $i(T)$, where $i$ is independent domination.

***Theorem 18***: For any non-trivial tree T, with at least two cutvertices, then

$$r_{Lb}(T) \leq i(T).$$

***Proof*** : Let A = $\{ v_1, v_2, v_3, \ldots, v_n \} \subset V(G)$ be the set of all end vertices in $G$ and $V_1 = V(G) \quad A$. Suppose there exists a set of vertices $C \subseteq V_1$ such that $N(u) \cap N(w) \neq \quad \forall\ u, w \in C$ which covers all vertices in $G$. Then $\{C\}$ forms a minimal independent dominating set of $G$. Suppose a non-trivial tree has $n \geq 2$ cutvertices and $m$ non end edges. Then we consider the following cases.

**Case 1**. Assume $G$ has $n$ cutvertices such that all cutvertices are of different degrees. Then in Lb(G) each block is of different regular. Let $b_1, b_2, b_3, \ldots, b_n$ be the number of blocks in Lb(G), $|\{b_1, b_2, b_3, \ldots, b_n\}| = m \quad 1$. Hence $|m \quad 1| \leq |C|$ gives $r_{Lb}(T) \leq i(T)$.

**Case 2**. Assume $G$ has at least two cutvertices with same degree. Then consider a set $\{ c_1, c_2, c_3, \ldots, c_k \}$ number of cutvertices with $c_i = c_j$, $1 \leq i$, $j \leq n$. Hence the complete blocks $b_i = b_j$, $b_i, b_j \in$ Lb(T) such that $|\{ c_1, c_2, c_3, \ldots, c_k \}| \leq |C|$ gives again $r_{Lb}(T) \leq i(T)$.

**Case 3.** Assume every cutvertex of G has same degree. Then every block of Lb(G) is same regular. Let Lb(G) has $\{ b_1, b_2, b_3, \ldots, b_n \}$ blocks. Now we have the partition $F_1 = \{b_i\}$ and $F_2 = \{b_j\}$, $1 \leq i$, $j \leq n$. such that each $b_i$ and $b_j$ are vertex disjoint blocks which are complete. Hence $|\{F_1, F_2\}| \leq |C|$ gives $r_{Lb}(T) \leq i(T)$.

From case 1, 2 and 3, we have $r_{Lb}(T) \leq i(T)$.

**Conclusion**

We studied the property of our concept by applying to some standard graphs. We also establish the regular number of block line graph of some standard graphs. Further we develop the upper bound in terms of minimum edge independence number of G and vertices of G. We establish some properties of this graph. Also many results are sharp.

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