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INTERNATIONAL JOURNAL OF CURRENT RESEARCH

International Journal of Current Research
Vol. 8, Issue, 10, pp.40413-40423, October, 2016

ISSN: 0975-833X

RESEARCH ARTICLE

FRACTIONAL STOCHASTIC CONTROL INVOLVE FRACTIONAL ITO-LEVY PROCESSES WITH APPLICATIONS TO FINANCE

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ARTICLE INFO

Article History:
Received 19th July, 2016
Received in revised form
22nd August, 2016
Accepted 14th September, 2016
Published online 30th October, 2016

Key words:

Fractional stochastic differential equation, Fractional stratonovichstochastic differential equation, Stratonovich stochastic differential equation, Dynkin formula, Hamilton-Jacobi - Bellman equation.

ABSTRACT

The main method of optimal control of systems described by Ito-levy processes which is Dynamic programming and Hamilton-Jacobi-Bellman equation and which supported by three types of forms with examples.

Citation: Sameer Qasim Hasan and Gaeth Ali Salum, 2016. "Fractional stochastic control involve fractional ito-levy processes with applications to finance", *International Journal of Current Research*, 8, (10), 40413-40433.

1. INTRODUCTION

The review paper is based on papers such as (Peter, 1990), (Ganig *et al*., 2008) and (ksendal and Sulem, 2014), which is important tools to study optimal control of fractional stochastic optimal control with fractional Ito-levy processes, stronovichIto-levy processes and fractional stronovich Ito-levy processes in details. In section tow we give some concepts about fractional levy process, fractional stochastic differential equation with levy process, some examples and Dynkin Formula. In section three we present an example about a portfolio optimization problem in an fractional stochastic Ito-levy process, we recognize the method that is to find the optimal fractional stochastic optimal control which is the HJB equation. In section four we givedetails about stochastic differential equation for stronovich Ito-levy process, some examples and Dynkin Formula. In section five we present an example about a portfolio optimization problem in an stronovich stochastic Ito-levy process and study the HJB-equation to find the optimal controlfor performance functional with for differential equation ofstronovich stochastic Ito-levy. In section six we givedetails about fractional stochastic differential equation for fractional stronovich Ito-levy process, some examples and Dynkin Formula. In section seven we present an example about a portfolio optimization problem in an fractionalstronovich stochastic Ito-levy process, and study the HJB-equation to find the optimal control for performance functional with differential equation of stronovich stochastic Ito-levy.

2. Fractional Stochastic Calculus for Fractional ITO-Levy processes

In this section we give a brief survey of fractional stochastic calculus for fractional Ito-levy processes.

For more details we refer to Chapter 1 in (ϕksendal and Sulem, 2007).

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Definition (2.1), (ϕksendal and Sulem, 2007)

A Levy processes on a probability space (Ω,F,p) is a processes, η(t) ≡η(t,w) with the following proparties

(i) η(0)=0.
(ii) η has stationary, independent increments.
(iii)η is stochastically continuous.

The jump of n at time t is $\Delta\eta=\eta(t) - \eta(t-)$.

Remark (2.1)

One can prove that ηalways has a cadlag (i.e. left continuous with right sided limits).

The jump measure N((0,t), U) gives the number o jumps of ηup to time t with jump size in the set $U \subseteq R/\{0\}$. If we assume that $\overline{U} \subseteq R/\{0\}$, then it can be show that U contains only finitely many jumps in any finite time interval. The Levy measure v(.) of η is defined by

v(U)=E(N((0,t), U)), ………(1)

and N(dt,d£) is the differential notation of the random measure N((0,t), U). Intuitively, £ can be regarded as generic jump size . let Ñ(.) denote the compensated jump measure of η, defined by

Ñ(dt,d£)≡ N(dt,d£)−v(d£)dt. ……….(2)

Assume that the Levy process has the form

$\eta(t)=at+bB^{H}(t)+\int_0^t \int £\tilde{N}(dt,d£)$, ………(3)

where$B^{H}(t)$ is fractional Brownian motion , and a,b are constants.

More general, we study the fractional Ito-Levy processes, which are the processes of the form

$x(t)=x+\int_0^t a(x(s),u(s))ds+\int_0^t b(x(s),u(s))\, dB^{H}(s)+\int_0^t \int r(s,£,w)\tilde{N}(ds,d£)$, …….(4)

wherea and band are continuous functional define in a metric space k, r(t,£) are predictable processes (predictable w.r.t the filtration $\mathcal{F}_t$ generate by η(s),for s≤t).

In differential form
$dx(t)= a(x(t), u(t))dt+ b(x(t),u(t))\, dB^{H}(t)+\int r(t,£)\tilde{N}(dt,d£)$, ……..(5)

we now proceed to the fractional Ito formula for Ito-Levy processes let x(t) be an Ito-Levy process defined in equation (4). let f:(0,T) ×Rbe a $C^{1,2}$ function and put Y(t)=f(t,x(t)).

Then Y(t) is also an fractional Ito-Levy process with representation

$Y(t)=f(t,x(t_0))+\int_0^t \frac{\partial f(s,x(s))}{\partial x} a(x(s))ds+\int_0^t \frac{\partial f(s,x(s)}{\partial x} b(x(s))\, dB^{H}(s)+$

$\int_0^s \int_0^t \frac{\partial^2 f(s,x(s)}{\partial^2 x} \int_0^p D_q(a(x(s))dsb(x(s))\emptyset_H(P-q)dqdp+$

$\int_0^s \int_0^t \frac{\partial^2 f(s,x(s)}{\partial^2 x} \int_0^p D_q(b(x(s))\, dB^{H}(s)b(x(s))\emptyset_H(P-q)dqdp+$

$\int_0^s \int_0^p \frac{\partial^2 f(s,x(s)}{\partial^2 x} b(x(q))\, b(x(p))\emptyset_H(P-q)dqdp+$

$\int\{f(s,x(s)+ r(s,£))-f(s,x(s))\}\tilde{N}(dt,d£)+\int\{f(s,x(s)+ r(s,£))-f(s,x(s)) -$

$\frac{\partial f(s,x(s)}{\partial x} r(x,£)\}v(d£)dt.$ ………(6)

Where the last term can be interpreted as the quadratic variation of jumps.

The Ito isometrics state the following

$E((\int_0^T \int r(s,£)\tilde{N}(dt,d£))^2)=E((\int_0^T \int r^2(s,£)v(d£)ds)$ ……….(7)

Example (2.1)

Suppose $\eta(t)=\int_0^t \int £\tilde{N}(ds,d£)$, we want to find the representation of $F=\eta^2(t)$. by the fractional Ito formula we get

$$\begin{aligned} d(\eta^2(t)) &= \int\{(\eta(t)+ £)^2-(\eta(t))^2\}\tilde{N}(ds,d£)+\int\{(\eta(t)+ £)^2-(\eta(t))^2-2\eta(t)\, £\}v(d£)dt \\ &= \int\{(\eta(t))^2+2\eta(t)\, £ +£^2-(\eta(t))^2\}\tilde{N}(ds,d£)+\int\{ £^2 v(d£)dt \\ &= \int\{2\eta(t)\, £\tilde{N}(ds,d£)\} + \int\{ £^2 v(d£)dt+\int\{ £^2\tilde{N}(ds,d£)\} \\ &= 2\eta(t)d\eta(t)+\int\{ £^2 v(d£)dt+\int\{ £^2\tilde{N}(ds,d£)\} \end{aligned} \quad ……..(8)$$

$$\eta^2(t) = T\int\{ £^2 v(d£)dt+\int_0^T(2\eta(t)d\eta(t)+\int_0^T \int\{ £^2\, \tilde{N}(ds,d£)\} \quad …….(9)$$

Note that it is not possible to write $F=\eta^2(T)$, as a constant + an integral w.r.t $d\eta(t)$

This has an interpretations in finance .it implies that in a normalized market with $\eta(t)$ as the risky asset price. The clime $\eta^2(t)$ is not replicable .this illustrates that markets based on le'vy processes are typically not complete.

Consider the following fractional stochastic differential equation (FSDE)

$$dx(t)=a(x(t))dt+ b(x(t))\, dB^H(t)+\int r(t,x(t^-),£)\tilde{N}(dt,d£)\quad x(0)=x, \quad ……(10)$$

here a, b and r are given functions. If these functions are Lipshitz continuouse with respect to x and with at most linear growth in x, uniformly in t, then a unique L^2-solution to the above FSDE exists.

Example (2.2):

The fractional Ito-Le'vy processes $x(t)$ is defined by

$$\begin{aligned} &dx(t)=x(t^-)(\, a(x(t))dt+ b(x(t))\, dB^H(t)+\int r(t,£)\tilde{N}(dt,d£)) \\ &x(0)=x, \end{aligned} \quad …....(11)$$

and $f(t,x(t))=Ln(x(t)$, then $x(t)=x\exp(f(t,x(t))$

$$\begin{aligned} x(t)=x\exp(&\int_0^t \frac{\partial f(s,x(s)}{\partial x} x(t^-)a(x(s),u(s))ds + \\ &\int_0^t \frac{\partial f(s,x(s)}{\partial x} x(t^-)b(x(s),u(s))\, dB^H(s)+ \\ &\int_0^s\int_0^t \frac{\partial^2 f(s,x(s)}{\partial^2 x}\int_0^p D_q(x(t^-)b(x(s),u(s))\, dB^H(s)+ \\ &\int_0^s\int_0^t \frac{\partial^2 f(s,x(s)}{\partial^2 x}\int_0^p D_q(x(s^-)b(x(s),u(s))\, dB^H(s)x(s^-)b(x(s),u(s))\emptyset_H(P-q)dqdp+ \\ &\int_0^s\int_0^p \frac{\partial^2 f(s,x(s)}{\partial^2 x} x(q^-)b(x(q),u(q))x(p^-)\, b(x(p),u(p))\emptyset_H(P-q)dqdp+ \\ &\int\{f(s,x(s)+ r(s,£))-f(s,x(s))\}\tilde{N}(dt,d£)+\int\{f(s,x(s)+ r(s,£))-f(s,x(s)) - \\ &\frac{\partial f(s,x(s)}{\partial x} r(x,£)\}v(d£)dt). \end{aligned} \quad ……(12)$$

Definition (2.2) (Ganig *et al.*, 2008)

The generator A^u of an Fractional Stochastic differential equation (5) defined by $A^u f = \frac{E(df(x(t)))}{dt}$ … ….(13)

Lemma (2.1):

If $x(t)$ defined in equation (5) and $f \in C_0^2(R)$, where C_0 corresponds to f having compact support , then $(A^u f)(x)$ exists for all x and

$$(A^u f)(x)=\frac{\partial f(s,x(t)}{\partial x} a(x(t)) +\int_0^s \frac{\partial^2 f(t,x(t))}{\partial^2 x}\int_0^p D_q(a(x(t))dtb(x(p))\emptyset_H(p-q)dqdp+\int_0^p \frac{\partial^2 f(p,x(p)}{\partial^2 x} b(x(q))\, b(x(p))\emptyset_H(p-q)dp$$
$$+\int\{f(s,x(s)+ r(s,£))-f(s,x(s)) - \frac{\partial f(s,x(s)}{\partial x} r(x,£)\}v(d£)dt \quad …….(14)$$

Lemma,"TheDynkin Formula"(2.2):

Let x(t) defined in equation (5) and T be a stopping time, let $f \in C_0^2(R)$ and assume that $E(\int_0^T (A^u f)(x) < \infty$, then by stochastic Taylor formula

$$f(t,x(t))=f(t,x(t_0))+\int_0^t \frac{\partial f(s,x(s)}{\partial x} a(x(s),u(s))ds +\int_0^t \frac{\partial f(s,x(s)}{\partial x} b(x(s),u(s))\, dB^H(s)+$$

$\int_0^s \int_0^t \frac{\partial^2 f(s,x(s)}{\partial^2 x} \int_0^p D_q(a(x(s),u(s))ds b(x(s),u(s))\emptyset_H(P-q)dqdp+$
$\int_0^s \int_0^t \frac{\partial^2 f(s,x(s)}{\partial^2 x} \int_0^p D_q(b(x(s),u(s))\, dB^H(s) b(x(s),u(s))\emptyset_H(P-q)dqdp+$
$\int_0^s \int_0^p \frac{\partial^2 f(s,x(s)}{\partial^2 x} b(x(q),u(q))\, b(x(p),u(p))\emptyset_H(P-q)dqdp+$
$\int\{f(s,x(s)+ r(s,£))-f(s,x(s))\}\tilde{N}(dt,d£)+\int\{f(s,x(s)+ r(s,£))-f(s,x(s)) -$
$\frac{\partial f(s,x(s)}{\partial x} r(x,£)\}v(d£)dt.$

By taking the expectation and truncate the series we have the following Dynkin formula

$E(f(t,x(t)))=f(t,x(t_0))+E(\int_0^t (A^u f)(x)dt,$ …….(15)

3.Fractional Stochastic Control

We start by motivating example as follows:

Example (3.1):

Suppose we have a financial market with two investment possibilities

(i)A risk-free asset with unit price $S_0(t)=1$.
(ii)A risk asset with unite price S(t)at time t given by
$dS(t)=S(t^-)(a(x(t))dt+ b(x(t))\, dB^H(t)+\int r(t,£)\tilde{N}(dt,d£)),$ ……..(16)

let u(t) denote a portfolio representing the fraction of the total wealth invested in the risky asset at time t. if we assume that u(t) is self –function , the corresponding wealth $x(t)=x_u(t)$ satisfy the state equation

$dx(t)=x(t^-)u(t)(a(x(t))dt+ b(x(t))\, dB^H(t)+\int r(t,£)\tilde{N}(dt,d£)),$ ……..(17)

the problem is to maximize $E(U(x_u(t)))$ over all $u\in A$, where$A$denotes the set of all admissible portfolios andU is a given utility function.

This is a special case of the following general stochastic control problem

$dY(t)= a(Y(t))dt+ b(Y(t))\, dB^H(t)+\int r(Y(t),£)\tilde{N}(dt,d£))$

$Y(0)=y$ ……...(18)

The performance functional is given by

$h_u(y)=E(\int_0^{Ts} f(Y(s),u(s))ds+g(Y(Ts))),$ ..……(19)

Where Ts=inf{t≥0}(bankruptcy time)

The problem is to find $u^* \in A$and $\varphi(y)=\sup_{u\in A} h_u(y)=h_{u^*}(y)$.

Theorem (3.1)"HJB equation ":

Suppose we can find a function $f \in C_0^2(R)$ such that

(i) $(A^u f)(y)+h(y,u)\leq 0$,$\forall u\in U$ where U is a set of possible control values and $(A^u f)(y)$ defined in (14)
(ii) $\lim_{t\to Ts}(f(y(t))=g(y(Ts))$

Then $f(y) \geq \varphi(y)$.

Proof

Using the (Dynkin Formula) to get that

$E(f(y(Ts))=f(y)+E(\int_0^{Ts} (A^u f)(y))$ ……(20)

By using (i) and (ii) in equation (20)

$E(g(y(Ts)) \leq f(y) - E(\int_0^{Ts} h(y,u)dt).$ ……...(21)

This implies

$f(y) \geq E(\int_0^{Ts} h(y,u)dt + g(y(Ts))).$

$= h_{u^*}(y), \forall u^* \in A,$ ………(22)

Which is mean that

$f(y) \geq \sup_{u \in A} h_u(y) = \varphi(y).$ ….(23)

To illustrate this result let us return to the optimal portfolio problem of Example (3.1)

Example (3.2):

Let U(x)=ln(x). then the problem is to maximize $E(\ln(x_u(T)))$. Put

$$dy(t)=\begin{bmatrix} dt \\ dx(t) \end{bmatrix}=\begin{bmatrix} 0 \\ x(t)u(t)\alpha(t) \end{bmatrix}dt+\begin{bmatrix} 0 \\ x(t)u(t)\beta(t) \end{bmatrix}dB^H(t)+\begin{bmatrix} 0 \\ x(t)u(t) \end{bmatrix}\int r(t,£)\tilde{N}(dt,d£). \quad …..(24)$$

and

$$(A^u f)(x)= u(t)\alpha(t)+\int_0^t \frac{-1}{x^2(t)}x(q)u(q)\,\beta(q)x(p)u(p)\,\beta(p)\emptyset_H(P-q)dqdp+\int\{\ln(x(t)+x(t)u(t)\,r(t,£))-\ln(x(t))-u(t)\,r(t,£)\}v(d£)dt. \quad (25)$$

By (HJB-equation)

$(A^u f)(x)+f(x)=0$, where $f(x)=U(x)=\ln(x)$

$$u(t)\alpha(t)+\int_0^p \frac{-1}{x^2(p)}x(q)u(q)\,\beta(q)x(p)u(p)\,\beta(p)\emptyset_H(P-q)dqdp+\int\{\ln(x(t)+x(t)u(t)\,r(t,£))-\ln(x(t))-u(t)\,r(t,£)\}v(d£)dt+\ln(x(t))=0 \quad …….(26)$$

in particular if $v(d£)=0$, then

$$u(t)=\frac{\int_0^p \frac{-1}{x^2(p)}x(q)u(q)\beta(q)x(p)u(p)\beta(p)\emptyset_H(P-q)dqdp-\ln(x)}{\alpha(t)}, \quad ……..(27)$$

is optimal control for the fractional Ito-Levy process for equation (24).

4. Stratonovich Stochastic Differential Equation

In this section we will study the stratonovich Ito-Levy processes, which are the processes of the form

$$x(t)=x+\int_0^t \tilde{a}(x(s))ds+\int_0^t b(x(s))\circ dB(t)+\int_0^t \int r(s,£,w)\tilde{N}(ds,d£), \quad ……...(28)$$

where $\tilde{a}(x(t))=a(x(t))-\frac{1}{2}b(x(t))\frac{\partial b(x(t))}{\partial x(t)}$, ……...(29)

$$b(x(t))\circ dB(t)=b(x(t))dB(t)+\frac{1}{2}b(x(t))\frac{\partial b(x(t))}{\partial x(t)}, \quad …….(30)$$

wherea and b are continuous functional define on a metric space k,B(t) is Brownian motionand r(t,£) are predictable processes (predictable w.r.t the filtration $\mathcal{F}_t$ generate by η(s),for s≤t).

In differential form

$$dx(t)=\tilde{a}(x(t))dt+b(x(t))\circ dB(t)+\int r(t,£)\tilde{N}(dt,d£), \quad ……..(31)$$

we now proceed to thestratonovich Ito formula for Ito-Levy processes let x(t) be an Ito-Levy process defined in equation (28). let f:(0,T) ×R be a $C^{1,2}$ function and put Y(t)=f(t,x(t)).

Then Y(t) is also an Ito-Levy process with representation

$$Y(t)=f(t,x(t_0))+\int_0^t \tilde{a}(x(t))\frac{\partial f(t,x(t))}{\partial x}dt+\int_0^t b(x(t))\frac{\partial f(t,x(t))}{\partial x}\circ dB(t)+$$

$\int\{f(s,x(s)+r(s,£))-f(s,x(s))\}\tilde{N}(dt,d£)+\int\{f(s,x(s)+r(s,£))-f(s,x(s))-\frac{\partial f(s,x(s)}{\partial x}r(x,£)\}v(d£)dt.$ ……...(32)

Example (4.1):

Suppose $\eta(t)=\int_0^t\int £\tilde{N}(ds,d£)$, we want to find the representation of $F=\eta^2(t)$. by the Ito formula we get

$$d(\eta^2(t))=\int\{(\eta(t)+£)^2-(\eta(t))^2\}\tilde{N}(ds,d£)+\int\{(\eta(t)+£)^2-(\eta(t))^2-2\eta(t)£\}v(d£)dt$$
$$=\int\{(\eta(t))^2+2\eta(t)£+£^2-(\eta(t))^2\}\tilde{N}(ds,d£)+\int\{£^2v(d£)dt$$
$$=\int\{2\eta(t)£\tilde{N}(ds,d£)\}+\int\{£^2v(d£)dt+\int\{£^2\tilde{N}(ds,d£)\}$$
$$=2\eta(t)d\eta(t)+\int\{£^2v(d£)dt+\int\{£^2\tilde{N}(ds,d£)\}$$
$$\eta^2(t)=T\int\{£^2v(d£)dt+\int_0^T(2\eta(t)d\eta(t)+\int_0^T\int\{£^2\tilde{N}(ds,d£)\}$$

Note that it is not possible to write $F=\eta^2(T)$, as a constant + an integral w.r.t $d\eta(t)$

This has an interpretations in finance .it implies that in a normalized market with $\eta(t)$ as the risky asset price. The clime $\eta^2(t)$ is not replicable .this illustrates that markets based on le'vy processes are typically not complete.

Consider the following fractional stratonovic stochastic differential equation (SDE)

$$dx(t)=\tilde{a}(x(t))dt+b(x(t))\circ dB(t)+\int r(t,x(t^-),£)\tilde{N}(dt,d£)\quad x(0)=x, \qquad \ldots\ldots\ldots(33)$$

here a, b and r are given functional. If these functions are Lipshitzcontinuouse with respect to x and with at most linear growth in x, uniformly in t, then a unique L^2-solution to the above SDE exists.

Example (4.2):

The geometric Ito-Le'vy processes x(t) is defined by

$$dx(t)=x(t^-)(\tilde{a}(x(t))dt+b(x(t))\circ dB(t)+\int r(t,£)\tilde{N}(dt,d£))$$
$$x(0)=x, \qquad \ldots\ldots\ldots\ldots(34)$$

and $f(t,x(t))=Ln(x(t)$, then $x(t)=x\exp(f(t,x(t))$

$x(t)=x\exp(\int_0^t\tilde{a}(x(t))\frac{\partial f(t,x(t))}{\partial x}dt+\int_0^t b(x(t))\frac{\partial f(t,x(t))}{\partial x}\circ dB(t)$
$\int\{f(s,x(s)+r(s,£))-f(s,x(s))\}\tilde{N}(dt,d£)+\int\{f(s,x(s)+r(s,£))-f(s,x(s))$
$-\frac{\partial f(s,x(s)}{\partial x}r(x,£)\}v(d£)dt).$ ………(35)

Lemma (4.3):

If x(t) defined in equation (28) and $f\in C_0^2(R)$, where C_0 corresponds to f having compact support , then from definition (2.2) $(A^u f)(x)$ exists for all x and

$$(A^u f)(x)=\tilde{a}(x(t))\frac{\partial f(t,x(t))}{\partial x}-\frac{1}{2}b(x(t))\frac{\partial b(x(t))}{\partial x}\frac{\partial f(t,x(t))}{\partial x}+$$

$$\frac{1}{2}b(x(t))\frac{\partial f(x(t))}{\partial x}\frac{\partial b(x(t))}{\partial x}+$$

$$+\int\{f(s,x(s)+r(s,£))-f(s,x(s))-\frac{\partial f(s,x(s)}{\partial x}r(x,£)\}v(d£)dt. \qquad \ldots\ldots\ldots(36)$$

Lemma ,"The Dynkin Formula" (4.4):

Let x(t) defined in equation (28) and T be a stopping time, let $f\in C_0^2(R)$ and assume that $E(\int_0^T(A^u f)(x)<\infty$, then by stochastic Taylor formula

$f(t,x(t))=f(t,x(t_0))+\int_0^t\tilde{a}(x(t))\frac{\partial f(t,x(t))}{\partial x}dt+\int_0^t b(x(t))\frac{\partial f(t,x(t))}{\partial x}\circ dB(t)+$
$\int\{f(s,x(s)+r(s,£))-f(s,x(s))\}\tilde{N}(dt,d£)+\int\{f(s,x(s)+r(s,£))-f(s,x(s))-$
$\frac{\partial f(s,x(s)}{\partial x}r(x,£)\}v(d£)dt.$ …......(37)

By taking the expectation and truncate the series we have the following Dynkin formula

$E(f(t,x(t)))=f(t,x(t_0))+E(\int_0^t (A^u f)(x)dt,$ ……(38)

5. Stratonovich Stochastic Control

We start by motivating example

Example (5.1):

Suppose we have a financial market with two investment possibilities

(i) A risk-free asset with unit price $S_0(t)=1$.
(ii) A risk asset with unite price $S(t)$ at time t given by

$dS(t)=S(t^-)(\tilde{a}(x(t))dt+ b(x(t)) \circ dB(t)+\int r(t,£)\tilde{N}(dt,d£)),$ …….(39)

let u(t) denote a portfolio representing the total wealth invested in the risky asset at time t. if we assume that u(t) is self –function, the corresponding wealth $x(t)=x_u(t)$ satisfy the state equation

$dx(t)=x(t^-)u(t)(\tilde{a}(x(t))dt+ b(x(t))\circ dB(t)+\int r(t,£)\tilde{N}(dt,d£)),$ …….(40)

the problem is to maximize $E(U(x_u(t)))$ over all $u\in A$, where Adenotes the set of all admissible portfolios and U is a given utility function.

This is a special case of the following general stochastic control problem

$dY(t)=\tilde{a}(Y(t))dt+ b(Y(t)) \circ dB(t)+\int r(Y(t),£)\tilde{N}(dt,d£))$

$Y(0)=y$ …….(41)

The performance functional is given by equation (19).

Theorem (5.2)"HJB equation ":

Suppose we can find a function $f \in C_0^2(R)$ such that

(i) $(A^u f)(y)+h(y,u)\leq 0$,$\forall u\in U$ where U is a set of possible control values and $(A^u f)(y)$
(ii) $\lim_{t\to Ts}(f(y(t))=g(y(Ts))$

Then $f(y) \geq \varphi(y)$.

Proof

Using the (Dynkin Formula) to get that
$E(f(y(Ts))=f(y)+E(\int_0^{Ts} (A^u f)(y))$ ….(42)

By using (i) and (ii) in (3.5)
$E(g(y(Ts)) \leq f(y)- E(\int_0^{Ts} h(y,u)dt).$ ….(43)

This implies
$f(y)\geq E(\int_0^{Ts} h(y,u)dt+g(y(Ts))$.
$=h_{u^*}(y), \forall u^* \in A,$ ….(44)

Which is mean that
$f(y) \geq \sup_{u\in A} h_u(y)= \varphi(y).$ …..(45)

To illustrate this result let us return to the optimal portfolio problem of Example (5.1)

Example (5.2):

Let $U(x)=\ln(x)$. then the problem is to maximize $E(\ln(x_u(T)))$. Put

$dy(t)=\begin{bmatrix} dt \\ dx(t) \end{bmatrix}=\begin{bmatrix} 1 \\ x(t)u(t)\alpha(t) \end{bmatrix}dt-\frac{1}{2}\begin{bmatrix} 0 \\ x(t)u(t)\beta(t) \end{bmatrix}\begin{bmatrix} 0 \\ u(t)\beta(t) \end{bmatrix}+\begin{bmatrix} 0 \\ x(t)u(t)\beta(t) \end{bmatrix}$
$\circ\, dB(t) +\begin{bmatrix} 0 \\ x(t)u(t) \end{bmatrix}\int r(t,£)\tilde{N}(dt,d£).$ ……..(46)

and

$(A^u f)(x)=\begin{bmatrix} 1 \\ u(t)\alpha(t) \end{bmatrix}-\frac{1}{2x(t)}\begin{bmatrix} 0 \\ x(t)u(t)\beta(t) \end{bmatrix}\begin{bmatrix} 0 \\ u(t)\beta(t) \end{bmatrix}+\int\{\ln(x(t)\, r(t,£))-\ln(x(t)r(t,£)\}v(d£)dt.$ ……..(47)

By (HJB-equation)

$(A^u f)(x)+f(x)=0$, where $f(x)=U(x)=\ln(x)$

$\begin{bmatrix} 1 \\ u(t)\alpha(t) \end{bmatrix}-\frac{1}{2x(t)}\begin{bmatrix} 0 \\ x(t)u(t)\beta(t) \end{bmatrix}\begin{bmatrix} 0 \\ u(t)\beta(t) \end{bmatrix}+\int\{\ln(x(t)\, r(t,£))-\ln(x(t)r(t,£)\}v(d£)dt+\ln(x(t))=0$ ……(48)

in particular if $v(d£)=0$, and take the drive the both sid

$u(t)=\frac{4x^2(t)\beta^2(t)}{\alpha(t)},$ …...(49)

is optimal control for the Ito-Levy process of equation (19).

6. StratonovichFractional Stochastic Differential Equation

In this section we will study the fractional stratonovich Ito-Levy processes, which are the processes of the form

$x(t)=x+\int_0^t \tilde{a}(x(s))ds+\int_0^t b(x(s))\circ dB^H(s)+\int_0^t\int r(s,£,w)\tilde{N}(ds,d£),$ ….(50)

where$\tilde{a}(x(t))=a(x(t))-\frac{1}{2}b(x(t))\,\frac{\partial b(x(t))}{\partial x(t)}$,

$b(x(t))\circ dB^H(t)=b(x(t))\, dB^H(t)+\frac{1}{2}b(x(t))\frac{\partial b(x(t))}{\partial x(t)},$

where a and b are continuous functional define in a metric space k,$B^H(t)$ is fractional Brownian motionand r(t,£) are predictable processes (predictable w.r.t the filtration F_t generate by η(s),for s ≤t).

In differential form

$dx(t)=\tilde{a}(x(t))dt+ b(x(t))\circ dB^H(t)+\int r(t,£)\tilde{N}(dt,d£),$ …..(51)

we now proceed to the fractional stratonovich Ito formula for Ito-Levy processes let x(t) be an Ito-Levy process defined in equation (50). let f:(0,T) ×R be a $C^{1,2}$ function and put Y(t)=f(t,x(t)).

Then Y(t) is also an fractional Ito-Levy process with representation

$Y(t)=f(t,x(t_0))+\int_0^t \frac{\partial f(s,x(s))}{\partial x}a(x(s))ds +\int_0^t \frac{\partial f(s,x(s))}{\partial x}b(x(s))\circ dB^H(s)+$
$\int_0^s\int_0^t \frac{\partial^2 f(s,x(s))}{\partial^2 x}\int_0^p D_q(a(x(s))ds\, b(x(s))\emptyset_H(P-q)dqdp+$
$\int_0^s\int_0^t \frac{\partial^2 f(s,x(s))}{\partial^2 x}\int_0^p D_q(b(x(s))\circ dB^H(s)b(x(s))\emptyset_H(P-q)dqdp+$
$\int_0^s\int_0^p \frac{\partial^2 f(s,x(s)}{\partial^2 x}b(x(q))\, b(x(p))\emptyset_H(P-q)dqdp+$
$\int\{f(s,x(s)+ r(s,£))-f(s,x(s))\}\tilde{N}(dt,d£)+\int\{f(s,x(s)+ r(s,£))-f(s,x(s)) - \frac{\partial f(s,x(s)}{\partial x}r(x,£)\}v(d£)dt.$ ……(52)

Example (6.1):

Suppose $\eta(t)=\int_0^t\int £\tilde{N}(ds,d£)$, we want to find the representation of F=$\eta^2(t)$. by the fractional Ito formula we get

$d(\eta^2(t))=\int\{(\eta(t)+ £)^2-(\eta(t))^2\}\tilde{N}(ds,d£)+\int\{(\eta(t)+ £)^2-(\eta(t))^2-2\eta(t)\, £\}v(d£)dt$
$= \int\{(\eta(t))^2+2\eta(t)\, £ +£^2-(\eta(t))^2\}\tilde{N}(ds,d£)+\int\{£^2 v(d£)dt$
$=\int\{2\eta(t)\, £\tilde{N}(ds,d£)\} + \int\{£^2 v(d£)dt+\int\{£^2\tilde{N}(ds,d£)\}$
$=2\eta(t)d\eta(t)+\int\{£^2 v(d£)dt+\int\{£^2\tilde{N}(ds,d£)\}$

$\eta^2(t) = T\int\{ £^2 v(d£)dt + \int_0^T (2\eta(t)d\eta(t) + \int_0^T \int\{ £^2\, \tilde{N}(ds, d£)\}$

Note that it is not possible to write F=$\eta^2(T)$, as a constant + an integral w.r.t dη(t)

This has an interpretations in finance .it implies that in a normalized market with η(t) as the risky asset price. The clime $\eta^2(t)$ is not replicable .this illustrates that markets based on le'vy processes are typically not complete.

Consider the following fractional stratonovic stochastic differential equation (FSDE)

$$dx(t) = \tilde{a}(x(t))dt + b(x(t))\circ dB^H(t) + \int r(t, x(t^-), £)\tilde{N}(dt, d£)\, x(0) = x, \quad \text{.....(53)}$$

here a, b and r are given functions. If these functions are Lipshitz continuouse with respect to x and with at most linear growth in x, uniformly in t, then a unique L^2-solution to the above FSDE exists.

Example(6.2):

The geometric fractional Ito-Le'vy processes x(t) is defined by

$$dx(t) = x(t^-)(\tilde{a}(x(t))dt + b(x(t))\circ dB^H(t) + \int r(t, £)\tilde{N}(dt, d£))$$

$$x(0) = x, \quad \text{......(54)}$$

and f(t,x(t))=Ln(x(t), then x(t)=x exp(f(t,x(t))

$$x(t) = x\exp(f(t, x(t_0)) + \int_0^t \frac{\partial f(s,x(s))}{\partial x}\tilde{a}(x(s))ds + \int_0^t \frac{\partial f(s,x(s))}{\partial x} b(x(s), u(s))\circ dB^H(s) + \int_0^s\int_0^t \frac{\partial^2 f(s,x(s))}{\partial^2 x}\int_0^p D_q(a(x(s))ds\, b(x(s))\emptyset_H(P-q)dqdp + \int_0^s\int_0^t \frac{\partial^2 f(s,x(s))}{\partial^2 x}\int_0^p D_q(b(x(s))\circ dB^H(s)b(x(s))\emptyset_H(P-q)dqdp + \int_0^s\int_0^p \frac{\partial^2 f(s,x(s)}{\partial^2 x} b(x(q))\, b(x(p))\emptyset_H(P-q)dqdp\; \int\{f(s, x(s) + r(s,£)) - f(s, x(s))\}\tilde{N}(dt, d£) + \int\{f(s, x(s) + r(s,£)) - f(s, x(s)) - \frac{\partial f(s,x(s)}{\partial x} r(x,£)\}v(d£)dt). \quad \text{.......(55)}$$

Lemma(6.1):

If x(t) defined in equation (50) and f ∈ $C_0^2(R)$, where C_0 corresponds to f having compact support , then $(A^u f)(x)$ exists for all x and

$$(A^u f)(x) = \frac{df(t,x(t))}{dt} + \frac{df(t,x(t))}{dx}\tilde{a}(x(t), u(t)) + \int_0^t \frac{d^2 f(s,x(s))}{d\,\partial^2 x}\int_0^p D_q(a(x(s))ds\, b(x(p))\emptyset_H(P-q)dqdp + \int_0^p \frac{d^2 f(p,x(p))}{d^2 x} b(x(q))b(x(p))\emptyset_H(P-q)dqdp + \int\{f(s, x(s) + r(s,£)) - f(s, x(s)) - \frac{df(s,x(s)}{dx} r(x,£)\}v(d£)dt. \quad \text{......(56)}$$

Lemma,"TheDynkin Formula" (6.2):

Let x(t) defined in equation (50) and T be a stopping time, let f ∈ $C_0^2(R)$ and assume that $E(\int_0^T (A^u f)(x) < \infty$, then by fractional stratonovich stochastic Taylor formula

$$f(t, x(t)) = f(t, x(t_0)) + \int_0^t \frac{\partial f(s,x(s))}{\partial x}\tilde{a}(x(s))ds + \int_0^t \frac{\partial f(s,x(s))}{\partial x} b(x(s))\circ dB^H(s) + \int_0^s\int_0^t \frac{\partial^2 f(s,x(s))}{\partial^2 x}\int_0^p D_q(\tilde{a}(x(s))ds\, b(x(s))\emptyset_H(P-q)dqdp + \int_0^s\int_0^t \frac{\partial^2 f(s,x(s)}{\partial^2 x}\int_0^p D_q(b(x(s))\circ dB^H(s)b(x(s))\emptyset_H(P-q)dqdp + \int_0^s\int_0^p \frac{\partial^2 f(s,x(s)}{\partial^2 x} b(x(q))\, b(x(p))\emptyset_H(P-q)dqdp + \int\{f(s, x(s) + r(s,£)) - f(s, x(s))\}\tilde{N}(dt, d£) + \int\{f(s, x(s) + r(s,£)) - f(s, x(s)) - \frac{\partial f(s,x(s)}{\partial x} r(x,£)\}v(d£)dt.$$

By taking the expectation and truncate the series we have the following Dynkin formula

$$E(f(t, x(t))) = f(t, x(t_0)) + E(\int_0^t (A^u f)(x)dt,$$

7. Fractional Stratonovich Stochastic Control

We start by motivating example

Example (7.1):

Suppose we have a financial market with two investment possibilities

(i) A risk-free asset with unit price $S_0(t)=1$.
(ii) A risk asset with unite price $S(t)$ at time t given by

$$dS(t)=S(t^-)(a(x(t),u(t))dt+b(x(t),u(t))\circ dB^H(t)+\int r(t,£)\tilde{N}(dt,d£)), \qquad \ldots\ldots\ldots.(57)$$

let u(t) denote a portfolio representing the fraction of the total wealth invested in the risky asset at time t. if we assume that u(t) is self –function , the corresponding wealth $x(t)=x_u(t)$ satisfy the state equation

$$dx(t)=x(t^-)u(t)(a(x(t))dt+b(x(t))\,dB^H(t)+\int r(t,£)\tilde{N}(dt,d£)), \qquad \ldots\ldots\ldots(58)$$

the problem is to maximize $E(U(x_u(t)))$ over all $u\in A$, where Adenotes the set of all admissible portfolios and U is a given utility function.

This is a special case of the following general stochastic control problem
$$dY(t)=a(Y(t))dt+b(Y(t))\circ dB^H(t)+\int r(Y(t),£)\tilde{N}(dt,d£)$$

$$Y(0)=y \qquad \ldots\ldots\ldots.(59)$$

The performance functional is given by equation (19)

Theorem (7.1)"HJB equation ":

Suppose we can find a function $f \in C_0^2(R)$ such that

(i) $(A^u f)(y)+h(y,u)\leq 0$,$\forall u\in U$ where U is a set of possible control values
(ii) $\lim_{t\to Ts}(f(y(t))=g(y(Ts))$

Then $f(y) \geq \quad (y)$.

Proof

Using the (Dynkin Formula) to get that

$$E(f(y(Ts))=f(y)+E(\int_0^{Ts}(A^u f)(y)) \qquad \ldots\ldots\ldots(60)$$

By using (i) and (ii) in (7.5)

$$E(g(y(Ts)) \leq f(y)-E(\int_0^{Ts} h(y,u)dt). \qquad \ldots\ldots\ldots(61)$$

This implies

$$f(y)\geq E(\int_0^{Ts} h(y,u)dt+g(y(Ts)).$$

$$=h_{u^*}(y),\ \forall u^* \in A, \qquad \ldots\ldots\ldots(62)$$

Which is mean that

$$f(y) \geq \sup_{u\in} \ h_u(y)= \quad (y). \qquad \ldots\ldots\ldots(63)$$

To illustrate this result let us return to the optimal portfolio problem of Example (7.1)

Example (7.2):

Let $U(x)=\ln(x)$. then the problem is to maximize $E(\ln(x_u(T)))$. Put

$$dy(t)=\begin{bmatrix} dt \\ dx(t) \end{bmatrix}=\begin{bmatrix} 0 \\ x(t)u(t)\alpha(t) \end{bmatrix}dt+\begin{bmatrix} 0 \\ x(t)u(t)\beta(t) \end{bmatrix}dB^H(t)+\begin{bmatrix} 0 \\ x(t)u(t) \end{bmatrix}\int r(t,£)$$

$\tilde{N}(dt,d£)$. ……….(64)

And

$(A^u f)(x)=\frac{dk(t)}{dt}+u(t)\alpha(t)-\frac{1}{2}x(t)u^2(t)\beta^2(t)-p\frac{1}{x^2(t)}u(q)\beta(q)u(p)\beta(q)+\int\{\ln(x(t)+r(t,£))-\ln(x(t))-u(t)\,r(t,£)\}v(d£)dt$. ……...(65)

By (HJB-equation)

$(A^u f)(x)+f(x)=0$, where $f(x)=f=\ln(x)+k(t)$

$\frac{dk(t)}{dt}+u(t)\alpha(t)-\frac{1}{2}x(t)u^2(t)\beta^2(t)-p\frac{1}{x^2(t)}u(q)\beta(q)u(p)\beta(q)+\int\{\ln(x(t)+r(t,£))-\ln(x(t))-u(t)\,r(t,£)\}v(d£)dt+\ln(x(t))+k(t)=0$.(66)

in particular if $v(d£)=0$ and take the derivative to both side , then

$u(t)=\frac{\alpha(t)}{x(t)\beta^2(t)}$ …….. (67)

is optimal control for the fractional Ito-Levy process for equation (64).

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