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RESEARCH ARTICLE

COEFFICIENT BOUNDS FOR A SUBCLASS OF BI-UNIVALENT FUNCTIONS USING NEW DIFFERENTIAL OPERATOR

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ABSTRACT

In this present paper, we introduce new subclasses $S_\Sigma(\gamma,\varphi)$ and $C_\Sigma(\gamma,\varphi)$ of bi-univalent functions defined in the open disk $U=\{z\in\mathbb{C}:|z|<1\}$. Furthermore, we find upper bounds for the second and third coefficients for functions in these new subclasses using differential operator.

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INTRODUCTION

Let A denote the class of functions $f(z)$ of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n, \tag{1.1}$$

which are analytic in the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$.

Further, by S we shall denote the class of functions $f\in A$ which are univalent in U. Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk U. However, the famous Koebe one-quarter theorem ensures that the image of the unit disk U under every function $f\in A$ contains a disk of radius $1/4$. Thus every univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z))=z$, $(z\in U)$

and

$$f\left(f^{-1}(w)\right)=w,\left(|w|<r_0(f),r_0(f)\geq\frac{1}{4}\right)$$

where

$$f^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\cdots. \tag{1.2}$$

A function $f\in A$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U.

We let Σ to denote the class of bi-univalent functions in U given by (1.1). If $f(z)$ is bi-univalent, it must be analytic in the boundary of the domain and such that it can be continued across the boundary of the domain so that $f^{-1}(z)$ is defined and analytic throughout $|w|<1$. Examples of functions in the class Σ are $\frac{z}{1-z},-log(1-z)$ and so on.

The coefficient estimate problem for the class S known as the Bieberbach conjecture, is settled by de-Branges [4], who proved that for a function $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$, in the class S, $|a_n|\leq n$, for $n=2,3,\cdots$, with equality only for the rotations of the Koebe function

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$$K_0(z)=\frac{z}{(1-z)^2}.$$

In 1967, Lewin [8] introduced the class Σ of bi-univalent functions and showed that $|a_2|<1.51$ for the functions belonging to Σ. It was earlier believed that for $f\in\Sigma$, the bound was $|a_n|<1$ for every n and the extremal function in the class was $\frac{z}{1-z}$. E.Netanyahu [10] in 1969, ruined this conjecture by proving that in the set Σ, $\max_{f\in\Sigma}|a_2|\le 4/3$. In 1969, Suffridge [13] gave an example of $f\in\Sigma$ for which $a_2=4/3$ and conjectured that $|a_2|\le 4/3.$ In 1981, Styer and Wright [12] disproved the conjecture that $|a_2|>4/3$. Brannan and Clunie [2] conjectured that $|a_2|\le\sqrt{2}$. Kedzierawski [6] in 1985 proved this conjecture for a special case when the function f and f^{-1} are starlike functions. Brannan and Clunie [2] conjectured that $|a_2|\le\sqrt{2}$.Tan [14] in proved that $|a_2|\le 1.485$ which is the best known estimate for functions in the class of bi-univalent functions.

Brannan and Taha [3] introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses $S^*(\alpha)$ and $C(\alpha)$ of the univalent function class Σ. Recently, Ali et al. [1] extended the results of Brannan and Taha [3] by generalising their classes using subordination.

An analytic function f is subordinate to an analytic function g ,written $f(z)\prec g(z)$, provided there is a Schwarz function w defined on U with $w(0)=0$ and $|w(z)|<1$ satisfying $f(z)=g(w(z))$. Ma and Minda [9], unified various subclasses of starlike and convex functions for which either of the quantity $\frac{zf'(z)}{f(z)}$ or $1+\frac{zf''(z)}{f'(z)}$ is subordinate to a more general superordinate function. For this purpose, they considered an analytic function ϕ with positive real part in the unit disk U, $\phi(0)=1$, $\phi'(0)>0$ and ϕ maps U onto a region starlike with respect to 1 and symmetric with respect to the real axis. Such a function has a series expansion of the form

$$\phi(z)=1+B_1z+B_2z^2+B_3z^3+\cdots,\qquad (B_1>0). \tag{1.3}$$

In this paper, for $f(z)\in\mathrm{A}$. Let a new differential operator be defined [5] on a class of analytic functions of the form (1.1) as follows:

$$F^0f(z)=f(z),$$

$$F^1f(z)=zf'(z)=:Ff(z)$$

and in general

$$F^nf(z)=F\left(F^{n-1}f(z)\right)\quad \left(n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}\right).$$

We easily find that

$$F^kf(z)=z+\sum_{n=2}^{\infty}C_{nk}a_nz^n\quad \left(n\in\mathbb{N}_0\right).$$

(1.4)

where $C_{nk}=\dfrac{n!}{|(n-k)|!}$

Definition: 1.1. Let γ be a non-zero complex number. A function $f(z)$ given by (1.1) is said to be in the class $\mathrm{S}_\Sigma(\gamma,\varphi)$ if the following conditions are satisfied:

$f\in\Sigma$ and

$$1+\frac{1}{\gamma}\left(\frac{z\left(F^jf(z)\right)'}{F^jf(z)}-1\right)\prec\varphi(z),\quad z\in\mathrm{U} \tag{1.5}$$

and

$$1+\frac{1}{\gamma}\left(\frac{w\left(F^jg(w)\right)'}{F^jg(w)}-1\right)\prec\varphi(w),\quad w\in\mathrm{U}, \tag{1.6}$$

where the function g is given by (1.2).

Definition: 1.2. Let γ be a non-zero complex number. A function $f(z)$ given by (1.1) is said to be in the class $\mathrm{C}_\Sigma(\gamma,\varphi)$ if the following conditions are satisfied:

$f\in\Sigma$ and

$$1+\frac{1}{\gamma}\left(\frac{z\left(F^jf(z)\right)''}{\left(F^jf(z)\right)'}\right)\prec\varphi(z),z\in\mathrm{U} \tag{1.7}$$

and

$$1+\frac{1}{\gamma}\left(\frac{w\left(F^jg(w)\right)''}{\left(F^jg(w)\right)'}\right)\prec\varphi(w),w\in\mathrm{U}, \tag{1.8}$$

where the function g is given by (1.2).

2. Coefficient estimates

Lemma: 2.1. [11] If $p\in\mathrm{P}$, then $|c_k|\le 2$ for each k, where P is the family of functions p analytic in U for which $\mathrm{Re}\,p(z)>0$, $p(z)=1+c_1z+c_2z^2+\cdots$ for $z\in\mathrm{U}$.

Theorem: 2.2. Let the function $f(z)\in\mathrm{A}$ be given by (1.1). If $f\in\mathrm{S}_\Sigma(\gamma,\varphi)$, then

$$|a_2|\le\frac{B_1\sqrt{B_1}\,|\gamma|}{\sqrt{\left|\left(2C_{3j}-C_{2j}^2\right)B_1^2\gamma+\left(B_1-B_2\right)C_{2j}^2\right|}}$$

and

$$|a_3|\le\frac{\left(B_1+|B_2-B_1|\right)|\gamma|}{2C_{3j}-C_{2j}^2}. \tag{2.1}$$

Proof: Since $f \in S_\Sigma(\gamma,\varphi)$, there exists two analytic functions $r,s:U\to U$, with $r(0)=0=s(0)$, such that

$$1+\frac{1}{\gamma}\left(\frac{z\left(F^j f(z)\right)'}{F^j f(z)}-1\right)=\varphi(r(z))$$

and

$$1+\frac{1}{\gamma}\left(\frac{w\left(F^j g(w)\right)'}{F^j g(w)}-1\right)=\varphi(s(z)). \tag{2.2}$$

Define the functions p and q by

$$p(z)=\frac{1+r(z)}{1-r(z)}=1+p_1z+p_2z^2+\cdots$$

and

$$q(z)=\frac{1+s(z)}{1-s(z)}=1+q_1z+q_2z^2+\cdots. \tag{2.3}$$

Or equivalently,

$$r(z)=\frac{p(z)-1}{p(z)+1}=\frac{1}{2}\left(p_1z+\left(p_2-\frac{p_1^2}{2}\right)z^2+\left(p_3+\frac{p_1}{2}\left(\frac{p_1^2}{2}-p_2\right)-\frac{p_1p_2}{2}\right)z^3+\cdots\right) \tag{2.4}$$

and

$$s(z)=\frac{q(z)-1}{q(z)+1}=\frac{1}{2}\left(q_1z+\left(q_2-\frac{q_1^2}{2}\right)z^2+\left(q_3+\frac{q_1}{2}\left(\frac{q_1^2}{2}-q_2\right)-\frac{q_1q_2}{2}\right)z^3+\cdots\right). \tag{2.5}$$

It is clear that p and q are analytic in U and $p(0)=1=q(0)$. Also p and q have positive real part in U and hence $|p_i|\le 2$ and $|q_i|\le 2.$

In the view of (2.3), (2.4) and (2.5), clearly,

$$1+\frac{1}{\gamma}\left(\frac{z\left(F^j f(z)\right)'}{F^j f(z)}-1\right)=\varphi\left(\frac{p(z)-1}{p(z)+1}\right)$$

and

$$1+\frac{1}{\gamma}\left(\frac{w\left(F^j g(w)\right)'}{F^j g(w)}-1\right)=\varphi\left(\frac{q(w)-1}{q(w)+1}\right). \tag{2.6}$$

Using (2.5) and (2.6) together with (1.3), one can easily verify that

$$\varphi\left(\frac{p(z)-1}{p(z)+1}\right)=1+\frac{B_1p_1}{2}z+\left(\frac{B_1}{2}\left(p_2-\frac{p_1^2}{2}\right)+\frac{1}{4}B_2p_1^2\right)z^2+\cdots \tag{2.7}$$

and

$$\varphi\left(\frac{q(w)-1}{q(w)+1}\right)=1+\frac{B_1q_1}{2}w+\left(\frac{B_1}{2}\left(q_2-\frac{q_1^2}{2}\right)+\frac{B_2q_1^2}{4}\right)w^2+\cdots. \tag{2.8}$$

Since $f\in\Sigma$ has the Maclaurin series given by (1.1), computation shows that its inverse $g=f^{-1}$ has the expansion given by (1.2). It follows from (2.6), (2.7) and (2.8) that

$$C_{2j}a_2=\frac{1}{2}B_1p_1\gamma, \tag{2.9}$$

$$2C_{3j}a_3-C_{2j}^2a_2^2=\frac{1}{2}\gamma B_1\left(p_2-\frac{1}{2}p_1^2\right)+\frac{1}{4}\gamma B_2p_1^2 \tag{2.10}$$

and

$$-C_{2j}a_2=\frac{1}{2}B_1\gamma q_1, \tag{2.11}$$

$$\left(4C_{3j}-C_{2j}^2\right)a_2^2-2C_{3j}a_3=\frac{1}{2}\gamma B_1\left(q_2-\frac{1}{2}q_1^2\right)+\frac{1}{4}\gamma B_2q_1^2. \tag{2.12}$$

From (2.9) and (2.11), it follows that

$$p_1=-q_1. \tag{2.13}$$

Now (2.10), (2.12) and (2.13) gives

$$a_2^2=\frac{B_1^3(p_2+q_2)\gamma}{4\left(\left(2C_{3j}-C_{2j}^2\right)B_1^2\gamma+C_{2j}^2(B_1-B_2)\right)}. \tag{2.14}$$

Using the fact that $|p_2|\le 2$ and $|q_2|\le 2$ gives the desired estimate on $|a_2|$,

$$|a_2|\le\frac{B_1\sqrt{B_1}|\gamma|}{\sqrt{\left|\left(2C_{3j}-C_{2j}^2\right)B_1^2\gamma+(B_1-B_2)C_{2j}^2\right|}}.$$

From (2.10)-(2.12), gives

$$a_3=\frac{\frac{\gamma B_1}{2}\left(\left(4C_{3j}-C_{2j}^2\right)p_2+C_{2j}^2q_2\right)+C_{3j}p_1^2(B_2-B_1)\gamma}{4\left(2C_{3j}^2-C_{3j}C_{2j}^2\right)}.$$

Using the inequalities $|p_1|\le 2$, $|p_2|\le 2$ and $|q_2|\le 2$ for functions with positive real part yields the desired estimation of $|a_3|$.

For a choice of $\varphi(z)=\frac{1+Az}{1+Bz}$, $-1\le B<A\le 1$, we have the following

Corollary: 2.3. Let $-1\le B<A\le 1$. If $f\in S_\Sigma\left(\gamma,\frac{1+Az}{1+Bz}\right)$, then

$$|a_2|\le\frac{|\gamma|(A-B)}{\sqrt{\left|\left(2C_{3j}-C_{2j}^2\right)(A-B)\gamma+(1+B)C_{2j}^2\right|}}$$

and

$$|a_3| \le \frac{|A-B|\left(1+|1+B|\right)|\gamma|}{\left(2C_{3j}-C_{2j}^2\right)}.$$

If we let $\varphi(z)=\left(\frac{1+z}{1-z}\right)^{\alpha}=1+2\alpha z+2\alpha^2 z^2+\cdots$, $0<\alpha\le 1$, in the above theorem, we get the following:

Corollary: 2.4. Let $0<\alpha\le 1$. If $f\in S_{\Sigma}(\gamma,\alpha)$, then

$$|a_2| \le \frac{|\gamma| 2\alpha}{\sqrt{\left|2\alpha\left(2C_{3j}-C_{2j}^2\right)\gamma+(1-\alpha)C_{2j}^2\right|}}$$

and

$$|a_3| \le \frac{\left(1+|\alpha-1|\right)2\alpha|\gamma|}{2C_{3j}-C_{2j}^2}.$$

Theorem: 2.5. Let the function $f(z)\in A$ be given by (1.1). If $f\in C_{\Sigma}(\gamma,\varphi)$, then

$$|a_2| \le \frac{B_1\sqrt{B_1}|\gamma|}{\sqrt{2\left|\left(3C_{3j}-2C_{2j}^2\right)B_1^2\gamma+2\left(B_1-B_2\right)C_{2j}^2\right|}}$$

and

$$|a_3| \le \frac{\left(B_1+|B_2-B_1|\right)|\gamma|}{2(3C_{3j}-2C_{2j}^2)}. \tag{2.15}$$

Proof: Since $f\in C_{\Sigma}(\gamma,\varphi)$, there exists two analytic functions $r,s:U\to U$, with $r(0)=0=s(0)$, such that

$$1+\frac{1}{\gamma}\left(\frac{z\left(F^j f(z)\right)''}{\left(F^j f(z)\right)'}\right)=\varphi(r(z))$$

and

$$1+\frac{1}{\gamma}\left(\frac{w\left(F^j g(w)\right)''}{\left(F^j g(w)\right)'}\right)=\varphi(s(z)). \tag{2.16}$$

Using (2.3), (2.4), (2.7) and (2.8), one can easily verified that

$$2C_{2j}a_2=\frac{1}{2}B_1 p_1\gamma, \tag{2.17}$$

$$6C_{3j}a_3-4C_{2j}^2a_2^2=\frac{1}{2}\gamma B_1\left(p_2-\frac{1}{2}p_1^2\right)+\frac{1}{4}\gamma B_2 p_1^2 \tag{2.18}$$

and

$$-2C_{2j}a_2=\frac{1}{2}B_1\gamma q_1, \tag{2.19}$$

$$\left(12C_{3j}-4C_{2j}^2\right)a_2^2-6C_{3j}a_3=\frac{1}{2}\gamma B_1\left(q_2-\frac{1}{2}q_1^2\right)+\frac{1}{4}\gamma B_2 q_1^2. \tag{2.20}$$

From (2.17) and (2.19), it follows that

$$p_1=-q_1. \tag{2.21}$$

Now (2.18), (2.20) and (2.21) gives

$$a_2^2=\frac{B_1^3\left(p_2+q_2\right)\gamma}{8\left(\left(3C_{3j}-2C_{2j}^2\right)B_1^2\gamma+2\left(B_1-B_2\right)C_{2j}^2\right)}. \tag{2.22}$$

Using the fact that $|p_2|\le 2$ and $|q_2|\le 2$ gives the desired estimate on $|a_2|$,

$$|a_2| \le \frac{B_1\sqrt{B_1}|\gamma|}{\sqrt{2\left|\left(3C_{3j}-2C_{2j}^2\right)B_1^2\gamma+2\left(B_1-B_2\right)C_{2j}^2\right|}}.$$

From (2.18)-(2.20), gives

$$a_3=\frac{\frac{\gamma B_1}{2}\left(\left(12C_{3j}^2-4C_{2j}^2\right)p_2+4C_{2j}^2q_2\right)+\left(B_2-B_1\right)\gamma p_1^2 3C_{3j}}{24C_{3j}\left(3C_{3j}-2C_{2j}^2\right)}.$$

Using the inequalities $|p_1|\le 2$, $|p_2|\le 2$ and $|q_2|\le 2$ for functions with positive real part yields

$$|a_3| \le \frac{\left(B_1+|B_2-B_1|\right)|\gamma|}{2\left(3C_{3j}-2C_{2j}^2\right)}.$$

For a choice of $\varphi(z)=\frac{1+Az}{1+Bz}$, $-1\le B<A\le 1$, we have the following corollary.

Corollary: 2.6. Let $-1\le B<A\le 1$. If $f\in S_{\Sigma}\left(\gamma,\frac{1+Az}{1+Bz}\right)$, then

$$|a_2| \le \frac{|\gamma|(A-B)}{\sqrt{2\left|\left(3C_{3j}-2C_{2j}^2\right)(A-B)\gamma+2(1+B)C_{2j}^2\right|}}$$

and

$$|a_3| \le \frac{|A-B|\left(1+|1+B|\right)|\gamma|}{2\left(3C_{3j}-2C_{2j}^2\right)}.$$

If we let $\varphi(z)=\left(\frac{1+z}{1-z}\right)^{\alpha}=1+2\alpha z+2\alpha^2 z^2+\cdots$, $0<\alpha\le 1$, in the above theorem, we get the following:

Corollary: 2.7. Let $0<\alpha\le 1$. If $f\in S_{\Sigma}(\gamma,\alpha)$, then

$$|a_2| \le \frac{|\gamma|\alpha}{\sqrt{\left|\left(3C_{3j}-2C_{2j}^2\right)\alpha\gamma+(1-\alpha)C_{2j}^2\right|}}$$

and

$$|a_3| \leq \frac{\left(1+|\alpha-1|\right)\alpha|\gamma|}{\left(3C_{3j}-2C_{2j}^{2}\right)}.$$

Remark: 2.1. If we let $\gamma = 1, j = 0$, Theorem: 2.2 and Theorem: 2.5 reduce to the result of R.M.Ali et.al [1], corollary 2.1 and corollary 2.2.

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