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International Journal of Current Research
Vol. 9, Issue, 07, pp.53489-53494, July, 2017

INTERNATIONAL JOURNAL OF CURRENT RESEARCH

ISSN: 0975-833X

RESEARCH ARTICLES

STUDY ON HYPERBOLICALLY SASAKIAN RECURRENT AND SYMMETRIC SPACES OF SECOND ORDER

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ARTICLE INFO

Article History:
Received 25th April, 2017
Received in revised form
29th May, 2017
Accepted 06th June, 2017
Published online 22nd July, 2017

Key words:

Sasakian Spaces,
Recurrent,
Symmetric.

ABSTRACT

Tachibana (1967) have studied on the Bochner Curvature tensor. Singh (1971-72) studied on Kaehlerian recurrent and Ricci-recurrent space of second order. Negi and Rawat (1999) studied some bi-recurrent and bi-symmetric properties in a Kaehlerian space. Further, Rawat, Kumar and Uniyal (2012) studied some bi-recurrent and bi-symmetric properties in hyperbolically Kaehlerian space.
In the present paper, we have been studied Hyperbolically Sasakian recurrent and symmetric space of second order. Several theorems also have been established and proved therein.

Citation: Rawat, K.S. and Sandeep Chauhan. 2017. "Study on hyperbolically Sasakian recurrent and symmetric spaces of second order", *International Journal of Current Research*, 9, (07), 53489-53494.

INTRODUCTION

Spaces with additional structures which arise in theoretical Physics play an important part in the theory of Riemannian spaces V_n . Such spaces are, in particular, "Classical" Kaehlerian and Sasakian spaces as well as hyperbolically Kaehlerian and Hyperbolically Sasakian spaces.

Definition (1.1): An odd-dimensional Riemannian space S_n is called a hyperbolically Sasakian space if, along with metric tensor g_{ij} , a complex structure tensor F_i^h satisfies the following conditions:

$$F_k^h F_i^k = \delta_i^h - X^h X_i , \tag{1.1}$$

$$F_k^h X^k = 0, \tag{1.2}$$

$$X^k X_k = 1, \tag{1.3}$$

$$g_{ki} F_j^k + g_{kj} F_i^k = 0, \tag{1.4}$$

$$F_{i,j}^h = X^h g_{ij} - \delta_j^h X_i , \tag{1.5}$$

Where $X_i = X^k g_{ki}$ is some vector.

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Differentiating (1.1), it is easy to establish that $F_i^h = X,_i^k$. This definition of Sasakian spaces is over determined.
The Riemannian curvature tensor field R_{ijk}^h is defined as

$$R_{ijk}^h = \partial_i \begin{Bmatrix} h \\ j \quad k \end{Bmatrix} - \partial_j \begin{Bmatrix} h \\ i \quad k \end{Bmatrix} + \begin{Bmatrix} h \\ i \quad \alpha \end{Bmatrix}\begin{Bmatrix} \alpha \\ j \quad k \end{Bmatrix} - \begin{Bmatrix} h \\ j \quad \alpha \end{Bmatrix}\begin{Bmatrix} \alpha \\ i \quad k \end{Bmatrix} ,$$

where $\partial_i = \frac{\partial}{\partial x^i}$ and $\{x^i\}$ denotes the real local coordinates.

The Ricci tensor and the Scalar curvature are respectively given by

$R_{ij} = R_{\alpha ij}^{\alpha}$ and $R = g^{ij} R_{ij}$

If we define a tensor S_{ij} by

$$S_{ij} = F_i^{\alpha} R_{\alpha j} , \tag{1.6}$$

Then, we have

$$S_{ij} = -S_{ji} , \tag{1.7}$$

$$F_i^{\alpha} S_{\alpha j} = -S_{i\alpha} F_j^{\alpha} , \tag{1.8}$$

And

$$F_i^{\alpha} S_{jk,\alpha} = R_{ji,k} - R_{ki,j} \tag{1.9}$$

It has been verified in Yano ([5]) pages 63, 68 that the metric tensor g_{ij} and the Ricci-tensor denoted by R_{ij} are hybrid in i and j. Therefore, we get

$$g_{ij} = g_{rs} F_i^r F_j^s , \tag{1.10}$$

And

$$R_{ij} = R_{is} F_i^r F_j^s , \tag{1.11}$$

The Holomorphically Projective curvature tensor P_{ijk}^h is given by

$$P_{ijk}^h = R_{ijk}^h + \frac{1}{(n+2)} \left(R_{ik}\, \delta_j^h - R_{jk} \delta_i^h + S_{ik} F_j^h - S_{jk} F_i^h + 2S_{ij} F_k^h\right) , \tag{1.12}$$

The Tachibana H-Concircular curvature tensor and the Weyl-Conformal curvature tensors are respectively give by

$$T_{ijk}^h = R_{ijk}^h + \frac{R}{n(n+2)} \left(g_{ik}\, \delta_j^h - g_{jk} \delta_i^h + F_{ik} F_j^h - F_{jk} F_i^h + 2F_{ij} F_k^h\right) , \tag{1.13}$$

and

$$C_{ijk}^h = R_{ijk}^h + \frac{1}{(n-2)} \left(R_{ik} \delta_j^h - R_{jk} \delta_i^h + g_{ik} R_j^h - g_{jk} R_i^h\right) - \frac{R}{(n-1)(n-2)} \left(g_{ik} \delta_j^h - g_{jk} \delta_i^h\right), \tag{1.14}$$

There is a Weyl-Concircular curvature tensor given by (Sinha, 1971)

$$Z_{ijk}^h = R_{ijk}^h + \frac{R}{n(n-1)} \left(g_{ik}\, \delta_j^h - g_{jk} \delta_i^h\right) , \tag{1.15}$$

If, we put

$$L_{ij} = R_{ij} - \frac{R}{n} g_{ij} \tag{1.16}$$

and

$$M_{ij} = F_i^{\alpha} L_{\alpha j} = S_{ij} - \frac{R}{n} F_i \tag{1.17}$$

Then from (1.12), (1.13), (1.16) and (1.17), we get

$$P^h_{ijk} = T^h_{ijk} + \frac{1}{(n+2)}\left(L_{ik}\delta^h_j - L_{jk}\delta^h_i + M_{ik}F^h_j - M_{jk}F^h_i + 2M_{ij}F^h_k\right), \tag{1.18}$$

and with the help of (1.14), (1.15), (1.16), and (1.17), we have

$$C^h_{ijk} = Z^h_{ijk} + \frac{1}{(n-2)}\left(L_{ik}\delta^h_j - L_{jk}\delta^h_i + g_{ik}L^h_j - g_{jk}L^h_i\right), \tag{1.19}$$

Now, we shall use the following:

Definition(1.3). A hyperbolically Sasakian space S_n is said to be recurrent space of second order, if we have

$$R^h_{ijk,ab} - \lambda_{ab}R^h_{ijk} = 0, \text{ or , equivalently } R_{ijkl,ab} - \lambda_{ab}R_{ijkl} = 0. \tag{1.20}$$

For some non-zero tensor field $\lambda_{ab,}$ and is known as recurrence tensor field.

A hyperbolically Sasakian space whose Ricci-tensor R_{ij} satisfies the equation

$$R_{ij,ab} - \lambda_{ab}R_{ij} = 0, \tag{1.21}$$

For some non-zero tensor $\lambda_{ab,}$ is called hyperbolically Sasakian Ricci-recurrent space of second order.

Multiplying the above equation by g^{ij}, we have

$$R_{,ab} - \lambda_{ab}R = 0. \tag{1.22}$$

Hyperbolically Sasakian Recurrent Spaces of second order

Definition(2.1). A hyperbolically Sasakian space satisfying the relation

$$P^h_{ijk,ab} - \lambda_{ab}P^h_{ijk} = 0, \text{ or, equivalently } P_{ijkl,ab} - \lambda_{ab}P_{ijkl} = 0. \tag{2.1}$$

For some non-zero tensor field λ_{ab} , will be called hyperbolically Sasakian projective recurrent space of second order.

Definition (2.2). A hyperbolically Sasakian space satisfying the relation

$$T^h_{ijk,ab} - \lambda_{ab}T^h_{ijk} = 0, \text{ or , equivalently } T_{ijkl,ab} - \lambda_{ab}T_{ijkl} = 0. \tag{2.2}$$

For some non-zero tensor field λ_{ab} , will be called hyperbolically Sasakian space with Tachibana H-Concircular recurrent curvature tensor of second order.

Definition (2.3). A hyperbolically Sasakian space satisfying the relation

$$C^h_{ijk,ab} - \lambda_{ab}C^h_{ijk} = 0, \text{ or, equivalently } C_{ijkl,ab} - \lambda_{ab}C_{ijkl} = 0. \tag{2.3}$$

For some non-zero tensor field λ_{ab} , will be called hyperbolically Sasakian space with Weyl-conformal recurrent curvature tensor of second order.

Definition (2.4). A hyperbolically Sasakian space satisfying the relation

$$Z^h_{ijk,ab} - \lambda_{ab}Z^h_{ijk} = 0, \text{ or , equivalently } Z_{ijkl,ab} - \lambda_{ab}Z_{ijkl} = 0. \tag{2.4}$$

For some non-zero recurrence tensor field λ_{ab} , will be called hyperbolically Sasakian space with Weyl-Concircular recurrent curvature tensor of second order.

Now, we have the following:

Theorem (2.1): If a hyperbolically Sasakian space satisfying any two of the following properties:

- the space is hyperbolically Sasakian Ricci-recurrent of second order,
- the space is hyperbolically Sasakian projective recurrent of second order,

- the space is hyperbolically Sasakian Tachibana H-Concircular recurrent of second order, then it must also satisfy the third.

Proof. Differentiating (1.18) covariantly w.r.to x^a , again differentiate the result thus obtained covariantly w.r.to x^b , we have

$$P^h_{ijk,ab} = T^h_{ijk,ab} + \frac{1}{(n+2)}\left(L_{ik,ab}\delta^h_j - L_{jk,ab}\delta^h_i + M_{ik,ab}F^h_j - M_{jk,ab}F^h_i + 2M_{ij,ab}F^h_k\right), \tag{2.5}$$

Multiplying (1.18) with λ_{ab} and subtracting the result thus obtained from (2.5), we have

$$P^h_{ijk,ab} - \lambda_{ab}P^h_{ijk} = T^h_{ijk,ab} - \lambda_{ab}T^h_{ijk} + \frac{1}{(n+2)}[(L_{ik,ab} - \lambda_{ab}L_{ik})\delta^h_j - (L_{jk,ab} - \lambda_{ab}L_{jk})\delta^h_i$$

$$+(M_{ik,ab} - \lambda_{ab}M_{ik})F^h_j - (M_{jk,ab} - \lambda_{ab}M_{jk})F^h_i + 2(M_{ij,b} - \lambda_{ab}M_{ij})F^h_k] \tag{2.6}$$

The statement of the above theorem follows in view of equations (1.21), (1.22), (2.1), (2.2), (1.16), (1.17) and (2.6).

Theorem(2.2). If a hyperbolically Sasakian space satisfying any two of the following properties:

- the space is hyperbolically Sasakian Ricci-recurrent of second order,
- the space is hyperbolically Sasakian space with Weyl-Conformal recurrent curvature tensor of second order,
- the space is hyperbolically Sasakian space with Weyl-Concircular recurrent curvature tensor of second order, then it must also satisfy the third.

Proof. A Hyperbolically Sasakian Ricci-recurrent space of second order, a Hyperbolically Sasakian space with Weyl-Conformal recurrent curvature tensor of second order and hyperbolically Sasakian space with Weyl-Concircular recurrent curvature tensor of second order are respectively characterized by the equations (1.21), (2.3) and (2.4).

Differentiating (1.19) covariantly w.r.to x^a , again differentiate the result thus obtained covariantly w.r.to x^b , we have

$$C^h_{ijk,ab} = Z^h_{ijk,ab} + \frac{1}{(n-2)}(L_{ik,ab}\delta^h_j - L_{jk,ab}\delta^h_i + g_{ik}L^h_{j,ab} - g_{jk}L^h_{i,ab}), \tag{2.7}$$

Multiplying (1.19) with λ_{ab} and subtracting the result thus obtained from (2.7), we have

$$C^h_{ijk,ab} - \lambda_{ab}C^h_{ijk} = Z^h_{ijk,ab} - \lambda_{ab}Z^h_{ijk} + \frac{1}{(n-2)}[(L_{ik,ab} - \lambda_{ab}L_{ik})\delta^h_j - (L_{jk,ab} - \lambda_{ab}L_{jk})\delta^h_i$$

$$+(L^h_{j,ab} - \lambda_{ab}L^h_j)g_{ik} - (L^h_{i,ab} - \lambda_{ab}L^h_i)\, g_{jk}], \tag{2.8}$$

The statement of the above theorem follows in view of (1.16), (1.17), (1.21), (2.3), (2.4) and (2.8).

Theorem (2.3). Every hyperbolically Sasakian recurrent space of second order is a hyperbolically Sasakian space with Tachibana H-Concircular recurrent space of second order.

Proof. Differentiating (1.13) covariantly w.r.to x^a, again differentiate the result thus obtained covariantly w. r .t. x^b, we have

$$T^h_{ijk,ab} = R^h_{ijk,ab} + \frac{R_{,ab}}{n(n+2)}(g_{ik}\delta^h_j - g_{jk}\delta^h_j + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k) \tag{2.9}$$

Multiplying (1.13) by λ_{ab} and subtracting the result thus obtained from (2.9), we have

$$T^h_{ijk,ab} - \lambda_{ab}T^h_{ijk} = R^h_{ijk,ab} - \lambda_{ab}R^h_{ijk} + \frac{(R_{,ab} - \lambda_{ab}R)}{n(n+2)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i + F_{ik}F^h_j - F_{jk}F^h_i + 2F_{ij}F^h_k), \tag{2.10}$$

Now, let the space be hyperbolically Sasakian recurrent space of second order, then equations (1.20), (1.21) and (1.22) are satisfied.

Making use of equations (1.20) and (1.21) in (2.10), we have

$$T^h_{ijk,ab} - \lambda_{ab}T^h_{ijk} = 0,$$

Which shows that the space is hyperbolically Sasakian space with Tachibana H-Concircular recurrent space of second order.

Hyperbolically Sasakian Symmetric Spaces of second order

Definition (3.1). A hyperbolically Sasakian space is said to be symmetric space of second order, if it satisfies the relation

$R^h_{ijk,ab} = 0,$ or, equivalently $R_{ijkl,ab} = 0,$ (3.1)

Obviously, a hyperbolically Sasakian symmetric space of second order is said to be hyperbolically Sasakian Ricci-symmetric space of second order, if

$R_{ij,ab} = 0,$ (3.2)

Multiplying the above equation by g^{ij} , we get

$R_{,ab} = 0,$ (3.3)

Definition (3.2). A hyperbolically Sasakian space satisfying the relation

$P^h_{ijk,ab} = 0,$ or, equivalently $P_{ijkl,ab} = 0,$ (3.4)

is called a hyperbolically Sasakian projective symmetric space of second order,

Definition (3.3). A hyperbolically Sasakian space satisfying the relation

$T^h_{ijk,ab} = 0,$ or, equivalently $T_{ijkl,ab} = 0,$ (3.5)

will be called hyperbolically Sasakian space with Tachibana H-Concircular symmetric space of second order.

Definition (3.4). A hyperbolically Sasakian space satisfying the relation

$C^h_{ijk,ab} = 0,$ or, equivalently $C_{ijkl,ab} = 0,$ (3.6)

will be called hyperbolically Sasakian space with Weyl-Conformal symmetric curvature tensor of second order.

Definition (3.5). A hyperbolically Sasakian space satisfying the relation

$Z^h_{ijk,ab} = 0,$ or, equivalently $Z_{ijkl,ab} = 0,$ (3.7)

is called hyperbolically Sasakian space with Weyl-Concircular symmetric curvature tensor of second order. Now, we have the following:

Theorem (3.1). If a hyperbolically Sasakian space satisfies any two of the following properties:

- The space is hyperbolically Sasakian Ricci-symmetric of second order,
- the space is hyperbolically Sasakian projective symmetric of second order,
- the space is hyperbolically Sasakian Tachibana H-Concircular symmetric of second order, then it must also satisfy the third.

Proof. A hyperbolically Sasakian Ricci-symmetric space of second order, a hyperbolically Sasakian projective symmetric space of second order and hyperbolically Sasakian space with Tachibana H-Concircular symmetric space of second order are respectively characterized by (3.2), (3.4) and (3.5).

The statement of the above theorem follows in view of (2.5), (3.2), (3.4) and (3.5).

Theorem (3.2). If a hyperbolically Sasakian space satisfies any two of the following properties:

- the space is hyperbolically Sasakian Ricci-symmetric of second order.
- the space is hyperbolically Sasakian With Weyl-Conformal symmetric curvature tensor of second order,
- the space is hyperbolically Sasakian space with Weyl-Concircular symmetric curvature tensor of second order, then it must also satisfy the third.

Proof. A hyperbolically Sasakian Ricci-symmetric space of second order, a Hyperbolically Sasakian space with Weyl-Conformal symmetric curvature tensor of second order and hyperbolically Sasakian space with Weyl-Concircular symmetric curvature tensor of second order are respectively characterized by (3.2), (3.6) and (3.7).

The statement of the above theorem follows in view of (2.7), (3.2), (3.6) and (3.7).

Theorem (3.3). Every hyperbolically Sasakian symmetric space of second order is a hyperbolically Sasakian space with Tachibana H-Concircular symmetric space of second order.

Proof. From (2.9), it follows that in a hyperbolically Sasakian symmetric space of second order, the Tachibana H-Concircular curvature tensor satisfies

$$T^h_{ijk,ab} = 0,$$

Which shows that the space is hyperbolically Sasakian space with Tachibana H-Concircular symmetric space of second order.

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