RESEARCH ARTICLE
SECONDARY UNITARY SIMILARITY OF MATRICES

1Krishnamoorthy, S. and 2Govindarasu, A.

1Professor and Head, Department of Mathematics, Ramanujan Research Centre, Government Arts College (Autonomous), Kumbakonam – 612 001, Tamil Nadu, India
2Department of Mathematics, AVC College, (Autonomous), Mannampandal, Mayiladuthurai, 609 305, Tamilnadu, India : Email: agavc@rediffmail.com

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ABSTRACT

The concept of s-unitarily similar matrices is introduced. Some theorem on s-unitarily similar matrices are given.

INTRODUCTION

The concept of s-unitary matrix was introduced in our earlier paper [4]. Also the concept of s-orthogonal similarity of real matrix is introduced by [5]. In this paper we define s-unitarily similar matrices and established some theorems on s-unitary matrices.

Preliminaries

Let $C_{n\times n}$ be the space of n$x$n complex matrices of order n. For $A \in C_{n\times n}$, let $A^T$, $\bar{A}$, $\bar{A}^s$, $(=A^b)$ denote transpose, conjugate, conjugate transpose, secondary transpose, conjugate secondary transpose of a matrix A respectively. Anna Lee [1] shown that for a complex matrix A, the usual transpose $A^T$ and secondary transpose $A^b$ are related as $A^b = VA^TV$ Where ‘V’ is the associated permutation matrix whose elements on the secondary diagonal are 1 and other elements are zero. We define $A^b = \bar{A}^s = (c_{ij})$ where $c_{ij} = \frac{a_{n-j+i, n-i+1}}{a_{n-i+1}}$ and $\bar{A}^s = VA^TV = A^b$. More over ‘V’ satisfies the following properties $V^T = \bar{V} = V^* = V$ and $V^2 = 1$

Secondary Unitary Similarity

Definition: 2.1 [4] Let $A \in C_{n\times n}$ A matrix A is called s-unitary if $A A^b = A^b A = I$

Definition: 2.2 [3] The n$x$n matrices A and B are called similar if there exist a nonsingular matrix S such that $A = S^{-1}BS$.

Definition: 2.3 [6] A and B are unitarily similar if there is a unitary matrix ‘U’ such that $A = U^*BU$

Now let us define s-unitarily similarity of two matrices A and B.

Definition: 2.4 Let $A, B \in C_{n\times n}$. A is said to be s-unitarily similar to B if there is a s-unitary matrix ‘U’ such that $A = U^bBU$

Theorem: 2.5 A is s-unitary iff every matrix s-unitarily similar to A, is s-unitary.

Proof

If $A$ is s-unitary then $A A^b = A^b A = I$.

B is any matrix which is s-unitarily similar to $A$.

$B = U^bAU$ where U is s-unitary.

$B^b = (U^bAU)^b = (U^bAU) A^b = U^bAU$

$U^b U = I$

$B^b = (U^bAU)(U^bAU) = U^bAU U^bAU = U^bAU$

$B = U^bAU$

$B^b = U^bAU U^bAU = U^bAU$

$U^b U = I$

$B = U^bAU = B^b= I$.

Coversely, if $B = U^bAU$ is s-unitary then $B^b = BB^b = U^bAU U^bAU = U^bAU$

$B = U^bAU$

$B^b = U^bAU U^bAU = U^bAU$

$B = U^bAU = B^b = I$

$B = U^bAU = B^b = I$.
Premultiplying by $U$ and Post multiplying by $U^\theta$ we get $A A^\theta = A^\theta A = I$. ∴ $A$ is s-unitary.

**Theorem: 2.6** Let $A \in C_{\text{max}}$. If $A$ is s-unitarily similar to a diagonal matrix which is $s$-unitary then $A$ is s-unitary.

**Proof**

$A$ is s-unitarily similar to a diagonal matrix $D$. Then there exists a $s$-unitary matrix $B$ such that $B^\theta AB = D$.

$A = BDB^\theta$

$A A^\theta = (BDB^\theta)(BDB^\theta)^\theta$ 

$A A^\theta = BD^2 B^\theta$ 

$A A^\theta = BB^\theta = I$ Since Band $D$ are s-unitary.

$A A^\theta = (BDB^\theta)(BDB^\theta)^\theta$ 

$A A^\theta = BD^2 B^\theta$ 

$A A^\theta = BB^\theta = I$.

$A A^\theta = A = I$. ∴ $A$ is s-unitary.

**Corollary 2.7:** Let $A$ and $B$ be s-unitarily similar. Then $A$ is s-unitary iff $B$ is s-unitary.

**Theorem 2.8** If $A$ and $B$ are s-unitarily similar

Then $\sum_{i,j} |b_{ij}|^2 = \sum_{i,j} |a_{ij}|^2$

**Proof**

We have $\sum_{i,j} |a_{ij}|^2 = \text{tr}(A^\theta A)$

$\sum_{i,j} |b_{ij}|^2 = \text{tr}(B^\theta B)$

$= \text{tr}((U^\theta AU)^\theta (U^\theta AU))$

$= \text{tr}(U^\theta (A^\theta A) U)$

$= \text{tr}(A^\theta A)$

$= \sum_{i,j} |a_{ij}|^2$

$\therefore \sum_{i,j} |b_{ij}|^2 = \sum_{i,j} |a_{ij}|^2$

**Definition 2.9** [2] Two families of $nxn$ matrices $\{A_1, A_2, \ldots, A_m\}$ and $\{B_1, B_2, \ldots, B_m\}$ are said to be unitarily similar if there exists a unitary matrix $U$ such that $U^\theta A_i U = B_i, \ i = 1,2,\ldots, m$

**Definition 2.10** Two families of $nxn$ matrices $\{A_1, A_2, \ldots, A_m\}$ and $\{B_1, B_2, \ldots, B_m\}$ are said to be $s$-unitarily similar if there exists a $s$-unitary matrix $U$ such that $U^\theta A_i U = B_i, \ i = 1,2,\ldots, m$

**Theorem 2.11** Let $A, B \in C_{\text{max}}$. If $A$ and $B$ are $s$-unitarily similar then the families $\{A, A^\theta\}$ and $\{B, B^\theta\}$ are $s$-unitarily similar.

**Proof**

$A$ and $B$ are $s$-unitarily similar $\Rightarrow B = U^\theta AU$ where $U$ is $s$-unitary matrix.

We have to show $B = U^\theta AU \ldots (i)$ and $B^\theta = U^\theta A^\theta U \ldots (ii)$

Proof (i) is obvious from definition of $s$-unitarily similarity.

(i) $\Rightarrow B = U^\theta AU$

$B^\theta = (U^\theta AU)^\theta = U^\theta A^\theta U$ which is (ii)

Therefore the families $\{A, A^\theta\}$ and $\{B, B^\theta\}$ are $s$-unitarily similar.

**Theorem 2.12** Let $A, B \in C_{\text{max}}$. If the families $\{A, A^\theta\}$ and $\{B, B^\theta\}$ are $s$-unitarily similar then the families $\{A, A^\theta\}$ and $\{B, B^\theta\}$ are $s$-similar.

**Proof**

Given $\{A, A^\theta\}$ and $\{B, B^\theta\}$ are $s$-unitarily similar

$B = U^\theta AU$ and $B^\theta = U^\theta A^\theta U$

We have to show $VB = U^{-1}VAU$ and $(VB)^\theta = U^{-1}(VA)^\theta U$

$B = U^\theta AU$ $VB = U^{-1}VU$ $AU = U^{-1}VAU$ Since $U^{-1}V = VU$

Therefore $VB$ is similar to $VA$.

$\Rightarrow B$ is $s$-similar to $A$.

Similarly we may prove $B^\theta$ is $s$-similar to $A^\theta$.

Therefore the families $\{A, A^\theta\}$ and $\{B, B^\theta\}$ are $s$-similar.

**Theorem 2.13** Let $A, B \in C_{\text{max}}$. If the families $\{A, A^\theta\}$ and $\{B, B^\theta\}$ are $s$-similar then $A$ and $B$ are $s$-unitarily similar.

**Proof**

Given $\{A, A^\theta\}$ and $\{B, B^\theta\}$ are $s$-similar.

$\Rightarrow P^{-1}VAP = VB$ and $P^{-1}(VA)^\theta P = (VB)^\theta$

$P^{-1}VAP = [P^{-1}(VA)P]P = PBP^{-1}$

$P^{-1}VAP = PBP^{-1}$

$VP = P(VA)^\theta P^{-1}$

$VA = VP(P^{-1})^\theta$

Since the $s$-hermitian $S$ which is a square root $PBP^{-1}$ can be represented as polynomial in $PBP^{-1}$ it follows from the relation above that $S(VA) = (VA)^\theta$.

$VB = P^{-1}(VA)P = (SU)^{-1}(VA)SU = U^{-1}S^{-1}VASU = U^{-1}VAU$ Therefore $VA$ is unitarily similar to $VB$.

$\Rightarrow A$ is $s$-unitarily similar to $B$.

**Definition 2.14:** $A \in C_{\text{max}}$ is called $s$-unitarily diagonalizable if there exist a $s$-unitary matrix ‘$U$’ for which $U^\theta AU$ is diagonal.

**Theorem 2.15** If $A \in C_{\text{max}}$ has eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$ counted according to multiplicity then $A$ is $s$-unitary implies $A$ is $s$-unitarily diagonalizable.

**Proof**

$A$ is $s$-unitary $\Rightarrow A$ is $s$-normal.

Then $A A^\theta$ is $s$-hermitian therefore $s$-unitarily diagonalizable.

Thus $U^\theta A^\theta AU = D = U^\theta AA^\theta U$.

Also $A, A^\theta, A A^\theta$ form a commuting family. This implies that eigen vectors of $A A^\theta$ are also eigen vectors of $A$. Since $A^\theta A$ has a complete orthonormal set we know that $U^\theta AU$ is also diagonal.

**REFERENCES**


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