RESEARCH ARTICLE

PROPERTIES OF ELECTRON-ION ACOUSTIC SOLITARY WAVES IN A MAGNETIZED DEGENERATE QUANTUM PLASMA

*1Promi Halder, 2Mukta, K. N. and 2Mamun, A. A.

1Department of Physics, Jagannath University, Dhaka-1100, Bangladesh
2Department of Physics, Jahangirnagar University, Savar, Dhaka-1342, Bangladesh

ABSTRACT

This research focuses on the "Quality of service The nonlinear propagation of electron-ion (El) acoustic solitary waves in a degenerate quantum plasma (containing relativistic magnetized quantum electrons and light ions in presence of stationary heavy ions) have been theoretically investigated. The Korteweg-de Vries (K-dV) and modified K-dV (mK-dV) equations are derived by adopting the reductive perturbation method. Their stationary solutions are derived and analyzed analytically as well as numerically to study some new basic features of the El acoustic solitary structures that are commonly found to exist in degenerate quantum plasma. It is found that the basic properties (viz. amplitude, width, and phase speed, etc.) of the El acoustic waves are significantly modified by the effects of relativistic ally degenerate electrons and light ions, quantum pressure, number densities of plasma particles, and external magnetic field, etc. The results of this theoretical investigation may be useful for understanding the formation and features of the solitary structures in astrophysical compact objects like white dwarfs, neutron stars, etc.

INTRODUCTION

The physics of dense quantum plasmas has received an immense interest not only because of their omnipresence in many astrophysical compact objects (like white dwarfs, neutron stars, active galactic nuclei, etc. (Woolsey et al., 2004; Shapiro, 1998; Garcia-Berro, 2010; Mamun, 2010)), but also for their available application in the laboratory i.e., intense laser-solid matter interaction experiment (Berezhiani, 1992; Murkhmd, 2006). The degenerate compact objects e.g. white dwarfs, neutron stars, etc. are usually consist of extremely dense iron/oxygen/carbon and helium nuclei (Shiikla, 2011; Massey, 1976) are ideal examples of degenerate plasma systems where the quantum mechanical effects (i.e., when the de Broglie thermal wavelength of the charged particles is equal or larger than the average inter-particle distance) play an important role in the plasma dynamics (Abdikian, 2016). The plasma in the interior of white dwarfs and in the crust of neutron stars are extremely dense and highly degenerate (Mamun, 2010; Chabrier, 2006; Lai, 2001; Harding, 2006) (plasma particles number densities can be of the order of $10^{20}$ cm$^{-3}$, or even more and the magnetic field strength can also be very large, i.e., $B \gg W^2G$). For such plasmas, the Fermi temperature $T_F$ is high and the quantum-mechanical effects associated with the quantum statistical pressure and the quantum force involving the Rohm potential (causing the plasma particles tunneling) are expected to play an important role (Manfredi, 2005). Therefore, by adding the quantum statistical pressure term (the Fermi-Dirac distribution) and the quantum diffraction term (the Bohm potential) to the fluid model generalized QHD model (Manfredi, 2001; Manfredi, 2005) is obtained. A number of works (Rouhani, 2014; Hossain, 2011; Haas, 2003; Bhowmik, 2007; Saeed-ur-Rehman, 2012; Hossen, 2015) have been done by considering the quantum effects of plasma particles in different plasma medium. The behaviour of high densities and low temperatures quantum plasmas was first studied by Pines (Pines, 1961). Haas et al. (Haas et al., 2003) found in their investigation that the Rohm potentials associated with the plasma particles significantly modify the basic features of the nonlinear IA waves. Bhowmik et al. (2007) investigated the effects of quantum diffraction parameter $\eta$, and the equilibrium density ratio of the plasma species in modifying the electron-acoustic (EA) waves in a quantum ET plasma.

*Corresponding author: Promi Halder

Department of Physics, Jagannath University, Dhaka-1100, Bangladesh.
On the other hand, Ali et al. (2007) analyzed the IA waves in an EPI plasma, and found that the nonlinear properties of the IA waves are significantly affected by the inclusion of the quantum terms in the momentum equations of electrons and positrons. But none of them considered the effects of external magnetic field and the presence of the heavy ions. It is well known that (Miller, 1987; Plastino, 1993; Gervino, 2012) the presence of external magnetic field (which causes the obliqueness of the wave propagation) plays a vital role in modifying the basic features of the linear and nonlinear waves in space and astrophysical plasmas (Mahmood, 2008; Sultana, 2010; El-Tantawy, 2012; Shahmansouri, 2013; Alinejad, 2013; Ashraf et al., 2014), and that the most of astrophysical degenerate quantum plasma systems like white dwarfs and neutron stars usually contain degenerate electrons and light ions along with heavy ions (Koester, 1990). This means that the effects of heavy ions and magnetic field must be considered, specially for the study of the nonlinear phenomena in the degenerate astrophysical objects (Woolsey, 2004; Shapiro, 1983; Torres et al., 2010; Mamun, 2010). Obliquely propagating electron-acoustic (EA) solitary waves (SWs) in a two electron population quantum rag-netoplasma was theoretically investigated by Masood and Mushtaq (2008). They found that propagation characteristics of the EA SWs are significantly affected by the presence of quantum corrections and the ratio of hot to cold electron concentration. Recently, Hossen and Mamun (Hossen, 2011c) have theoretically investigated the nonlinear positron-acoustic (PA) waves propagating in the fully relativistic electron-positron-ion plasma and found that the effects of relativistic degeneracy of electrons and positrons, static heavy ions, plasma particles velocity, and enthalpy, etc. have significantly modified the basic properties of the PA SWs. Using a fully relativistic set of two fluids plasma equations, Lee and Choi (Lee, 2007), Tribeche and Boukhalfa (2011), Saberian et al., (2011), and Akbari-Moghanjoughi (2011) have studied the characteristics of the nonlinear IA waves in different fully relativistic plasmas. Therefore, in our present work, we have examined the basic properties of the El acoustic waves propagating in a degenerate quantum plasma composed of relativistically magnetized quantum electrons and light ions in the presence of stationary heavy ions. We have considered the quantum mechanical (such as tunneling associated with the Bohm potential) effects for both electrons and light ions which abundantly occurs in different astrophysical situations (viz. white dwarfs, neutron stars, active galactic nuclei, etc. (Woolsey, 2004; Shapiro, 1983; Garcia-Berro, 2010; Mamun, 2010), and laboratory plasmas like intense laser-solid matter interaction experiment (S. 6). The manuscript is organized as in Sec. II, the basic equations governing our plasma model are presented; in Sec. III and IV, the K-dV and mK-dV equations along with their solutions are derived; in Sec. V, a brief discussion is given.

**MODEL EQUATIONS**

We consider a collision less plasma system with an ambient magnetic field directed along the z axis, i.e., (Bo = \( \hat{z}B_o \)). where \( \hat{z} \) is a unit vector in the z direction. The obliquely propagation of the nonlinear acoustic SWs through our considered plasma system is assumed such that the wave vector lies in the x-z plane. At equilibrium, the quasi-neutrality condition can be expressed as 
\[ Z_i\eta_{b_0} + \eta_{io} - \eta_{eo} = 0, \]
where \( \eta_{b_0}, \eta_{io}, \) and \( \eta_{eo} \) are the equilibrium number densities of immobile heavy ions, light ions, and electrons, respectively, and \( Z_0 \) is the immobile heavy ions charge state. The unnormalized dynamic equations for the considered magnetized quantum El plasmas are as follows:

\[ \frac{\partial N_s}{\partial T} + \nabla \cdot (N_s U_s) = 0, \]

\[ \frac{\partial U_s}{\partial T} + (U_s \cdot \nabla) U_s = \frac{q_s}{m_s} \left[ -\nabla \Phi + \frac{1}{c} \frac{1}{2} U_s \times B \right] 
- \frac{\nabla P^*_s}{m_s N_s} + \frac{\hbar^2}{2m_e} \nabla \left( \frac{1}{\sqrt{n_s}} \nabla \sqrt{N_s} \right), \]

\[ \nabla^2 \Phi = -4\pi e \left( N_i - N_e + Z_i N_h \right), \]

\[ P^*_s = \frac{\pi m^3 c^5}{3h^3} \left[ R_s \left( R_s^2 + 1 \right)^{1/2} \left( 2R_s^2 - 3 \right) + 3 \sinh^{-1}(R_s) \right], \]

in which the relativity parameter \( R_s = \frac{pF_s}{m_i c} \) and \( pF_s = \left( 3h^3 n_s / 8\pi \right)^{1/3} \) is the Fermi relativistic momentum of s’th species. Taking \( N_s = N_{s0} + N_{s1}, N_{s1} \) is the perturbation of the s’th species with \( |N_{s1}| << N_{s0} \) and the Taylor expansion up to second order, Eq. (4) turns to

\[ \frac{\partial N_s}{\partial T} + \nabla \cdot (N_s U_s) = 0, \]

\[ \frac{\partial U_s}{\partial T} + (U_s \cdot \nabla) U_s = \frac{q_s}{m_s} \left[ -\nabla \Phi + \frac{1}{c} \frac{1}{2} U_s \times B \right] 
- \frac{\nabla P^*_s}{m_s N_s} + \frac{\hbar^2}{2m_e} \nabla \left( \frac{1}{\sqrt{n_s}} \nabla \sqrt{N_s} \right), \]

\[ \nabla^2 \Phi = -4\pi e \left( N_i - N_e + Z_i N_h \right), \]

\[ P^*_s = \frac{\pi m^3 c^5}{3h^3} \left[ R_s \left( R_s^2 + 1 \right)^{1/2} \left( 2R_s^2 - 3 \right) + 3 \sinh^{-1}(R_s) \right], \]
the stretched coordinates (under consideration), we will de
To study the nonlinear propagation of the electro
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and the normalized equations for the electron quantum fluid can be written as

\[ P_s^* = P_{s0} + \frac{\partial P_s^*}{\partial N_s} N_{s1} + \frac{1}{2} \frac{\partial^2 P_s^*}{\partial N_s^2} N_{s1}^2, \]

\[ = P_{s0} + \frac{2 \epsilon F_s}{3 \gamma_{s0}} N_{s1} + \frac{(3 R_0^2 + 4) \epsilon F_s}{9 \gamma_{s0}^3 N_{s0}} N_{s1}^2, \]

(5)

where \( \gamma_{s0} = \sqrt{1 + \frac{R_0^2}{R_s^2}} \), \( R_s = P_{s0} / m_e c, \ P_{s0} = (3h^3 n_{s0} / 8\pi)^{1/3} \), and \( \epsilon F_s = h^2 (3\pi^2 n_s)^{2/3} / 2m_e \). In order to normalize the
the basic Eqs. (1)-(3), we use the following normalized parameters

\[ T \to \epsilon \omega_{pe} \nabla, \nabla \rightarrow \frac{\omega_{pe}}{uF_e} \nabla, \Phi \rightarrow \frac{e \phi}{2 \epsilon F_e}, U_s \rightarrow \frac{u_s}{uF_e} \]

\[ N_s \rightarrow \frac{n_s}{n_{s0}}, \omega_{pe} \rightarrow \frac{4\pi e^2 n_{s0}}{m_e}, H_s \rightarrow \frac{\hbar \omega_{pe}}{2 \epsilon F_e} \]

\[ uF_s \rightarrow \frac{2 \epsilon F_s}{m_e}, \Omega_c \rightarrow \frac{eB_c}{m_e c}, \Omega_e \rightarrow \frac{\omega}{\omega_{pe}} \]

\[ P_s^* \rightarrow \frac{P_s}{2 \epsilon F_e n_{s0}}, \eta \rightarrow \frac{\epsilon F_e}{\epsilon F_i} \sigma \rightarrow \frac{n_{m0}}{n_{e0}}. \]

(6)

The normalized equations for the ion quantum fluid can be written as

\[ \frac{\partial n_i}{\partial t} + \nabla (n_i u_i) = 0, \]

\[ \frac{\partial u_i}{\partial t} + (u_i \nabla) u_i = -\nabla \phi + \Omega_c (u_i \times \hat{z}) - \frac{\nabla P_i}{n_i} \]

\[ + \frac{H^2}{2(1 - \sigma \Omega_h)^{2/3}} \nabla \left( \frac{1}{\sqrt{n_i}} \nabla \sqrt{n_i} \right), \]

(8)

and the normalized equations for the electron quantum fluid can be written as

\[ \frac{\partial n_e}{\partial t} + \nabla (n_e u_e) = 0, \]

\[ \frac{\partial u_e}{\partial t} + (u_e \nabla) u_e = \nabla \phi - \Omega_c (u_e \times \hat{z}) - \frac{\eta \nabla P_e}{n_e} \]

\[ + \frac{H^2}{2} \nabla \left( \frac{1}{\sqrt{n_e}} \nabla \sqrt{n_p} \right), \]

(10)

and the normalized form of the Poisson equation is

\[ \nabla^2 \phi = n_e - n_i (1 - \sigma \Omega_h) - Z_n \sigma. \]

(11)

**K-dV EQUATION**

To study the nonlinear propagation of the electrostatic perturbation modes in the relativistically magnetized quantum plasmas (under consideration), we will derive the K-dV equation employing the reductive perturbation method (47). So, we first introduce the stretched coordinates (Washimi, 1966) as
\[ \xi = \varepsilon^{3/2} (l_y^2 + l_z^2 - V_p l_z), \]  
\[ \tau = \varepsilon^{3/2} l, \]  
where \( V_p = \omega / \kappa (\omega) \) is the wave phase speed, \( \varepsilon (0 < \varepsilon < 1) \) is a measure of the solitary wave amplitude i.e., a measure of the weakness of the dispersion or of the nonlinear effect, \( l_x, l_y, \) and \( l_z \) are the direction cosines of the wave vector \( \mathbf{k} \) along the \( x, y, \) and \( z \) axes, respectively, so that \( l_x^2 + l_y^2 = 1. \) We then expand the perturbed quantities \( n_i, n_e, u_{sx, sy}, u_{sz}, \) and \( \phi \) about their equilibrium values in power series of \( \varepsilon \) as

\[ n_i = 1 + \varepsilon n_i^{(1)} + \varepsilon^2 n_i^{(2)} + \varepsilon^3 n_i^{(3)} + \ldots \ldots \]  
(14)

\[ n_e = 1 + \varepsilon n_e^{(1)} + \varepsilon^2 n_e^{(2)} + \varepsilon^3 n_e^{(3)} + \ldots \ldots \]  
(15)

\[ u_{sx,y} = \varepsilon^{3/2} u_{sx,y}^{(1)} + \varepsilon^2 u_{sx,y}^{(2)} + \varepsilon^{5/2} u_{sx,y}^{(3)} + \ldots \ldots \]  
(16)

\[ u_{sz} = \varepsilon u_{sz}^{(1)} + \varepsilon^2 u_{sz}^{(2)} + \varepsilon^3 u_{sz}^{(3)} + \ldots \ldots \]  
(17)

\[ \phi = \varepsilon \phi^{(1)} + \varepsilon^2 \phi^{(2)} + \varepsilon^3 \phi^{(3)} + \ldots \ldots \]  
(18)

Now, substituting Eqs. (12)-(18) in Eqs. (7)-(11), and equating the coefficients for the lowest order of \( \varepsilon \), we obtain the first order continuity equations, the \( z \)-component of the momentum equations, and Poisson's equation, which after simplification, we can write as

\[ n_i^{(1)} = \frac{3 \gamma_{i0} l_z^2}{3 \gamma_{i0} V_p^2 - l_z^2} \phi^{(1)}, \]  
(19)

\[ u_{sz}^{(1)} = \frac{3 \gamma_{i0} l_z V_p}{3 \gamma_{i0} V_p^2 - l_z^2} \phi^{(1)}, \]  
(20)

\[ n_e^{(1)} = \frac{3 \gamma_{e0} l_z^2}{\eta l_z^2 - 3 \gamma_{e0} V_p^2} \phi^{(1)}, \]  
(21)

\[ u_{sz}^{(1)} = \frac{3 \gamma_{e0} l_z^2 V_p}{\eta l_z^2 - 3 \gamma_{e0} V_p^2} \phi^{(1)}, \]  
(22)

\[ V_p = \sqrt{\frac{\gamma_{e0} l_z^2 + \eta \gamma_{e0} l_z^2 (1 - Z_s \sigma)}{3 \gamma_{e0} l_z^2 Z_s \sigma}}, \]  
(23)

We define \( \delta = \cos \delta \), where \( \delta \) is the angle between the directions of the wave propagation vector \( \mathbf{k} \) and the external magnetic field \( \mathbf{B}_0 \). The Eq. (23) represents the dispersion relation for the acoustic type electrostatic waves in the degenerate quantum plasma under consideration. We can write the first order \( x \)- and \( y \)-components of the momentum equations as

\[ u_{xp}^{(1)} = \frac{3 \gamma_{i0} l_z^2 V_p^2}{\Omega_x (3 \gamma_{i0} V_p^2 - l_z^2)} \frac{\partial \phi^{(1)}}{\partial \xi}, \]  
(24)

\[ u_{yp}^{(1)} = \frac{3 \gamma_{i0} l_z^2 V_p^2}{\Omega_y (3 \gamma_{i0} V_p^2 - l_z^2)} \frac{\partial \phi^{(1)}}{\partial \eta}, \]  
(25)
Eqs. (24)-(27), represent the $x$— and $y$—components of the ion-electron electric field drift, respectively. Again, substituting Eqs. (12)-(18) in Eq. (8) and Eq. (10) and using Eqs. (19)-(27), we obtain the next higher order $x$- and $y$—components of the momentum equations as

$$u_{x}^{(1)} = -\frac{3l_{x} \gamma_{e0} V_{p}^{2}}{\Omega_{c} (3 \gamma_{e0} V_{p}^{2} - l_{x}^{2})} \frac{\partial \phi^{(1)}}{\partial \xi},$$

$$u_{y}^{(1)} = -\frac{3l_{y} \gamma_{e0} V_{p}^{2}}{\Omega_{c} (\eta l_{y}^{2} - 3 \gamma_{e0} V_{p}^{2})} \frac{\partial \phi^{(1)}}{\partial \xi},$$

$$u_{x}^{(2)} = \frac{3V_{p}^{3} \gamma_{e0} l_{x}}{\Omega_{c}^{2} (3 \gamma_{e0} V_{p}^{2} - l_{x}^{2})} \frac{\partial^{2} \phi^{(1)}}{\partial \xi^{2}},$$

$$u_{y}^{(2)} = \frac{3V_{p}^{3} \gamma_{e0} l_{y}}{\Omega_{c}^{2} (3 \gamma_{e0} V_{p}^{2} - l_{y}^{2})} \frac{\partial^{2} \phi^{(1)}}{\partial \xi^{2}},$$

$$u_{x}^{(2)} = -\frac{3V_{p}^{3} \gamma_{e0} l_{x}}{\Omega_{c}^{2} (\eta l_{y}^{2} - 3 \gamma_{e0} V_{p}^{2})} \frac{\partial^{2} \phi^{(1)}}{\partial \xi^{2}},$$

$$u_{y}^{(2)} = -\frac{3V_{p}^{3} \gamma_{e0} l_{y}}{\Omega_{c}^{2} (\eta l_{x}^{2} - 3 \gamma_{e0} V_{p}^{2})} \frac{\partial^{2} \phi^{(1)}}{\partial \xi^{2}},$$

Eqs. (28)-(31), denote the $y$— and $x$—components of the ion-electron polarization drift, respectively. Further, substituting Eqs. (12)-(18) in Eqs. (7)-(11), we obtain the next higher order continuity equations, the $z$- component of the momentum equations, and Poisson's equation, which can be given as

$$\frac{\partial n_{i}^{(0)}}{\partial \tau} - V_{p} \frac{\partial n_{i}^{(2)}}{\partial \xi} + l_{x} \frac{\partial u_{i}^{(2)}}{\partial \xi} + l_{y} \frac{\partial u_{i}^{(2)}}{\partial \xi} + l_{z} \frac{\partial u_{i}^{(2)}}{\partial \xi} + l_{x} \frac{\partial n_{i}^{(2)}}{\partial \xi} = 0,$$

$$l_{x} \frac{\partial}{\partial \xi} \left( n_{i}^{(1)} u_{i}^{(1)} \right) = 0,$$

$$\frac{\partial u_{i}^{(1)}}{\partial \tau} - V_{p} \frac{\partial u_{i}^{(2)}}{\partial \xi} + l_{x} u_{i}^{(0)} \frac{\partial u_{i}^{(2)}}{\partial \xi} + l_{y} \frac{\partial u_{i}^{(2)}}{\partial \xi} + l_{z} \frac{\partial u_{i}^{(2)}}{\partial \xi} + l_{x} \frac{\partial n_{i}^{(2)}}{\partial \xi} = 0,$$

$$l_{z} \frac{\partial}{\partial \xi} \left( n_{i}^{(1)} u_{i}^{(1)} \right) = 0,$$

$$-H^{2} l_{z} \frac{\partial^{3} n_{i}^{(1)}}{\partial \xi^{3}} = 0,$$

$$\frac{\partial n_{i}^{(1)}}{\partial \tau} - V_{p} \frac{\partial n_{i}^{(2)}}{\partial \xi} + l_{x} \frac{\partial n_{i}^{(2)}}{\partial \xi} + l_{y} \frac{\partial n_{i}^{(2)}}{\partial \xi} + l_{z} \frac{\partial n_{i}^{(2)}}{\partial \xi} + l_{x} \frac{\partial n_{i}^{(2)}}{\partial \xi} = 0,$$

$$l_{z} \frac{\partial}{\partial \xi} \left( n_{i}^{(1)} u_{i}^{(1)} \right) = 0,$$
where the amplitude solution of the SWs can be written as we introduce another stretched coordinates. Now, to investigate the properties of the El acoustic SWs, we are interested in the SWs solution of the K

\[
\frac{\partial^2 \phi^{(1)}}{\partial \zeta^2} = n_e^{(2)} - (1-Z_h \sigma)n_i^{(2)}
\]

(36)

Now, simplifying Eqs. (32)-(36) by using Eqs. (19)-(31), and combining each other, we finally obtain our desired equation in the form

\[
\frac{\partial^2 \phi^{(1)}}{\partial \tau} + A \frac{\partial \phi^{(1)}}{\partial \zeta} + B \frac{\partial^3 \phi^{(1)}}{\partial \zeta^3} = 0,
\]

(37)

Eq. (37) is the well known K-dV equation describing the dynamics of the SWs propagating in the degenerate quantum plasma. The nonlinear coefficient, \( A \) and the dispersion coefficient, \( B \) are given as

\[
A = \frac{L_1 + L_2 + L_3 + L_4}{6 \gamma_{e0} V_p^2 a^2 b + 6 \gamma_{e0} V_p a b^3 (1-Z_h \sigma)},
\]

(38)

\[
B = \frac{K_1 - K_2 - K_3 + K_4 - K_5}{72 \gamma_{e0} V_p l_z^2 a^2 \Omega_e^2 + 72 \gamma_{e0} V_p l_z^4 b^2 \Omega_e^2 (1-Z_h \sigma)},
\]

(39)

Where

\[
a = (3 \gamma_{e0} V_p^2 - l_z^2), \quad b = (\eta l_z^2 - 3 \gamma_{e0} V_p^2),
\]

\[
L_1 = 27 \gamma_{e0} V_p^2 a^2 l_z^2, \quad L_2 = \eta a^3 l_z^4 (3 \gamma_{e0}^2 - 3 \gamma_{e0} + 4),
\]

\[
L_3 = 27 \gamma_{e0} V_p b^3 l_z^2 (1-Z_h \sigma),
\]

\[
L_4 = b^3 l_z^4 (3 \gamma_{e0}^2 - 3 \gamma_{e0} + 4) (1-Z_h \sigma),
\]

\[
K_1 = 4 a^2 b^2 \Omega_e^2, \quad K_2 = 36 \gamma_{e0} V_p^4 a^2 (1-l_z^2),
\]

\[
K_3 = 9 \gamma_{e0} l_z^4 a^2 \Omega_e^2
\]

\[
K_4 = 36 \gamma_{e0} V_p^4 b^2 (1-l_z^2), \quad (1-Z_h \sigma),
\]

\[
K_5 = 9 \gamma_{e0} l_z^4 H^2 b^2 \Omega_e^2
\]

(40)

Now, to investigate the properties of the El acoustic SWs, we are interested in the SWs solution of the K-dV equation. To do so, we introduce another stretched coordinates. \( \zeta = \xi - u_0 \tau \). After the coordinate transformation, the steady state \( (\partial/\partial \tau = 0) \) solution of the SWs can be written as

\[
\phi^{(1)} = \phi_m \sec h^2 \left( \frac{\zeta}{\delta_1} \right)
\]

(41)

where the amplitude \( \phi_m = 3 U_0 / A \) and the width, \( \delta_1 = \sqrt{4 B / U_0} \). It is seen that the quantum parameter \( H \) is present in \( \delta_1 \), whereas \( \phi_m \) is totally independent of this parameter.

mK-DV EQUATION
The K-dV equation (Eq. (37)) is the result of the second order calculation in the smallness parameter $\varepsilon$, where the quadratic nature has been revealed by the nonlinear term $A \phi^{(1)} \frac{\partial \phi^{(1)}}{\partial \xi}$. For plasmas with more than two species as like our system, however, there can arise cases where $A$ vanishes at a particular value of a certain parameter $\sigma$, and Eq. (37) fails to describe nonlinear evolution of perturbation. So, higher order calculation is needed at this critical value $\sigma = \sigma_c$. For this reason, to derive the mK-dV equation, we apply the following stretched coordinates (Washimi, 1966)

\[ \xi = \varepsilon^{1/2}(l_x + l_y + l_z - V_p t), \]
\[ \tau = \varepsilon^{3/2} t, \]

We then expand the variables $n_i, n_e, u_{ix,y}, u_{iz}$, and $\phi$, in power series of $\varepsilon$ as

\[ n_i = 1 + \varepsilon^{1/2} n_i^{(1)} + \varepsilon^{3/2} n_i^{(3)} + ..., \]
\[ n_e = 1 + \varepsilon^{1/2} n_e^{(1)} + \varepsilon^{3/2} n_e^{(3)} + ..., \]
\[ u_{ix,y} = \varepsilon u_{ix,y}^{(1)} + \varepsilon^{3/2} u_{ix,y}^{(3)} + ..., \]
\[ u_{iz} = \varepsilon^{1/2} u_{iz}^{(1)} + \varepsilon^{3/2} u_{iz}^{(3)} + ..., \]
\[ \phi = \varepsilon^{1/2} \Phi^{(1)} + \varepsilon^{3/2} \Phi^{(3)} + ..., \]

By using Eq. (42)-(48) in Eqs. (7)-(11), we found the values of $n_i^{(1)}, n_e^{(1)}, u_{ix,y}^{(1)}, u_{iz}^{(1)}$, and $V_p$ as like as that of the K-dV equation. To the next higher order of $\varepsilon$, we obtain a set of equations, which, after using the values of $n_i^{(1)}, n_e^{(1)}, u_{ix,y}^{(1)}, u_{iz}^{(1)}$, and $V_p$ can be simplified as

\[ n_i^{(2)} = \frac{8 W_i^2 l_x^4 y_0^3}{2(3\gamma_{i0}V_p^2 - l_z^2)^3} \left[ \Phi^{(1)} \right]^2 \]
\[ + \frac{3l_x^3 y_0}{(3\gamma_{i0}V_p^2 - l_z^2)} \Phi^{(2)} \]
\[ n_e^{(2)} = -\frac{8 W_e^2 l_x^4 y_0^3}{2(3\gamma_{e0}V_p^2 - l_z^2)^3} \left[ \Phi^{(1)} \right]^2 \]
\[ + \frac{3l_x^3 y_0}{(3\gamma_{e0}V_p^2 - l_z^2)} \Phi^{(2)} \]
\[ u_{iz}^{(2)} = \frac{l_x l_z^2 (3\gamma_{i0}^2 - 3\gamma_{e0}^2 + 4)}{3\gamma_{i0} \gamma_{e0} V_p^2} \frac{\partial u_{iz}^{(2)}}{\partial \xi} + \frac{l_x^2 l_z (3\gamma_{i0}^2 - 3\gamma_{e0}^2 + 4)}{\gamma_{i0} \gamma_{e0} (3\gamma_{i0}V_p^2 - l_z^2)^2} \Phi^{(1)} \frac{\partial \Phi^{(1)}}{\partial \xi} \]
\[ + \frac{3V_p^2 l_x^2 y_0}{\gamma_{i0}^2 (3\gamma_{i0}V_p^2 - l_z^2)^2} \frac{\partial^2 \Phi^{(1)}}{\partial \xi^2}. \]
\[ u^{(2)}_{\xi \xi} = \frac{l_z}{3 \Omega \gamma_{\xi 0}} \frac{\partial n^{(2)}_{\xi}}{\partial \xi} - \frac{l_z}{3 \gamma_{\xi 0} \Omega_{\xi}} \frac{l_z (3 \gamma_{\xi 0}^2 - 3 \gamma_{\eta 0}^2 + 4)}{\Phi^{(1)}} \frac{\partial \Phi^{(1)}}{\partial \xi} \]

\[ + \frac{3 \eta l_z^2 \gamma_{\eta 0}^2}{\Omega_{\xi}^2 (3 \gamma_{\xi 0} V_p^2 - l_z^2)^2} \frac{\partial \Phi^{(1)}}{\partial \xi}, \]

\[ u^{(2)}_{\eta \eta} = \frac{n l_z}{3 \Omega \gamma_{\eta 0}} \frac{\partial n^{(2)}_{\eta}}{\partial \eta} - \frac{n l_z (3 \gamma_{\eta 0}^2 - 3 \gamma_{\xi 0}^2 + 4)}{\gamma_{\eta 0} \Omega_{\eta} (\eta l_z^2 - 3 \gamma_{\xi 0} V_p^2)^2} \frac{\partial \Phi^{(1)}}{\partial \eta} \frac{\partial \Phi^{(1)}}{\partial \eta} \]

\[ + \frac{3 \eta l_z^2 \gamma_{\eta 0}^2}{\Omega_{\eta}^2 (\eta l_z^2 - 3 \gamma_{\xi 0} V_p^2)^2} \frac{\partial \Phi^{(1)}}{\partial \eta}, \]

\[ u^{(2)}_{\nu} = \frac{9 V_p^2 l_z^2 \gamma_{\nu 0}^3 + l_z^3 (3 \gamma_{\xi 0}^2 - 3 \gamma_{\eta 0}^2 + 4)}{\gamma_{\xi 0} (3 \gamma_{\xi 0} V_p^2 - l_z^2)^2} \Phi^{(1)} \frac{\partial \Phi^{(1)}}{\partial \xi} \]

\[ + \frac{l_z}{V_p} \Phi^{(1)} + \frac{n l_z}{3 V_p \gamma_{\xi 0}} n^{(2)}_{\xi}, \]

\[ \rho^{(2)} = -\frac{1}{2} A \left\{ \Phi^{(1)} \right\} = 0, \]

where

\[ A_1 = \left[ \frac{T_1 + T_2}{(n l_z^2 - 3 \gamma_{\xi 0} V_p^2)} \right] + \frac{(1 - Z \sigma) (P_1 + P_2)}{3 (3 \gamma_{\xi 0} V_p^2 - l_z^2)}, \]

Where

\[ T_1 = 8 V_p^2 l_z^3 \gamma_{\nu 0}^3, \quad T_2 = 3 n l_z^3 (3 \gamma_{\xi 0}^2 - 3 \gamma_{\eta 0}^2 + 4) \]

\[ P_1 = 8 V_p^2 l_z^3 \gamma_{\xi 0}^3, \quad P_2 = 3 n l_z^3 (3 \gamma_{\xi 0}^2 - 3 \gamma_{\eta 0}^2 + 4) \]

To the next higher order of \( \Xi \), we obtain a set of equations which after simplification as follows

\[ \frac{\partial n^{(3)}_{\xi}}{\partial \xi} = \frac{18 V_p^2 l_z^2 \gamma_{\xi 0}^3}{(3 \gamma_{\xi 0} V_p^2 - l_z^2)^2} \frac{\partial \Phi^{(1)}}{\partial \xi} + \frac{3 l_z^2 \gamma_{\xi 0}}{(3 \gamma_{\xi 0} V_p^2 - l_z^2)} \frac{\partial \Phi^{(3)}}{\partial \xi} \]
Now, combining Eqs. (60) after simplifying by using Eqs. (49)-(57), we obtain the well-known mK-dV equation as follows
\[
\frac{\partial \Phi^{(1)}}{\partial \tau} + C \left( \frac{\partial \Phi^{(1)}}{\partial \xi} \right) + B \frac{\partial^3 \Phi^{(1)}}{\partial \xi^3} = 0,
\]

(64)

where

\[
C = \frac{a^2 b^2}{18V^2 \rho \gamma_{\sigma}^2 \gamma_{\sigma}^2 (1 - Z_{h} \sigma)} \left[ \frac{K_1}{2b^2 \gamma_{\sigma}} - \frac{K_2}{2b^2} \frac{G_4 + G_5}{Z_{h}} \left( 1 - Z_{h} \sigma \right) \right]
\]

(65)

Where

\[
K_1 = G_1 + G_2 + G_3 + G_4, \quad K_2 = G_5 + G_6 - G_7,
\]

\[
R_1 = F_1 + F_2 + F_3 + F_4, \quad R_2 = F_5 + F_6 - F_7.
\]

(64)

Now, taking the same stretching as like the solution of the K-dV SWs, the stationary SWs solution of Eq. (64) can be directly given as

\[
\Phi^{(1)} = \Phi_{m} \sec \left( \frac{\xi}{\delta_{2}} \right),
\]

(67)

where the amplitude, \( \Phi_{m} = \sqrt{6U_{\sigma}/C} \) and the width, \( \delta_{2} = \sqrt{B/U_{\sigma}} \), where \( \delta_{2} \) depends \( H \). On the other hand, \( \Phi_{m} \) is totally independent of these parameter.

**DISCUSSION**

The properties of the acoustic SWs in degenerate quantum plasmas are discussed in this section numerically. The effects of quantum diffraction are related not only in degenerate astrophysical plasmas but also may be meaningful in some dense laboratory plasma containing electrons and light ions, in presence of stationary heavy ions.

We emphasize the effects of the propagation angle, quantum pressure, relativistic factors, immobile heavy ions charge state, and densities of the plasma components on the acoustic SWs in such degenerate quantum plasmas. For our purpose, we have derived the K-dV and

![FIG. 1: Showing the variation of \( V_{\rho} \) with \( \sigma \) for \( U_{\sigma} = 0.01 \), \( \delta = 40 \), \( \gamma_{e0} = 1.33 \), \( \gamma_{i0} = 1.29 \) and \( Z_{b} = 2 \).

The dotted line for \( \eta = 1.1 \), the solid line for \( \eta = 1.2 \), and the dotdashed line for \( \eta = 1.3 \).]
FIG. 2: Showing the variation of $\Delta_1$ (K-dV) with $H$ for $U_0 = 0.01, \gamma_{c0} = 1.33, \gamma_{i0} = 1.29, \delta = 40, Z_h = 1, \eta = 1.1$, and $\sigma = 0.1$. The dotted line for $\Omega_c = 0.3$, the solid line for $\Omega_c = 0.4$, and the dotdashed line for $\Omega_c = 0.5$.

FIG. 3: Showing the effects of $\eta$ on the K-dV solitary profiles for $U_0 = 0.01, \gamma_{c0} = 1.33, \gamma_{i0} = 1.29, \sigma = 0.4, \delta = 40, \Omega_c = 0.4, Z_h = 1, H = 0.3, R_e = 2.5$, and $R_i = 2.2$. The lower red dotted line for $\eta = 1.1$, the lower blue solid line for $\eta = 1.13$, the upper green dotdashed line for $\eta = 1.16$, and the upper black solid line for $\eta = 1.19$.

FIG. 4: Showing the effects of $R_i$ on the (K-dV) SWs profiles for $U_0 = 0.01, \gamma_{c0} = 1.33, \gamma_{i0} = 1.29, \sigma = 0.4, \delta = 40, \Omega_c = 0.4, Z_h = 1, H = 0.3, R_e = 2.5$, and $\eta = 1.2$. The dotted line for $R_i = 1$, the solid line for $R_i = 2$, and the dotdashed line for $R_i = 3$. 
FIG. 5: Showing the effects of $\Omega_c$ on the mK-dV SWs profiles for $U_0 = 0.01$. $\gamma_{e0} = 1.33$, $\gamma_{i0} = 1.29$, $\sigma = 0.2$, $\delta = 60$, $Z_h = 1$, $H = 0.3$, $R_e = 2.5$, $R_i = 2.2$, and $\eta = 1.6$. The dotted line for $\Omega_c = 0.3$, the solid line for $\Omega_c = 0.4$, and the dotdashed line for $\Omega_c = 0.5$.

FIG. 6: Showing the effects of $\sigma$ on the mK-dV SWs profiles for $U_0 = 0.01$. $\gamma_{e0} = 1.33$, $\gamma_{i0} = 1.29$, $\Omega_c = 0.3$, $Z_h = 1$, $H = 0.3$, $R_e = 2.5$ and $R_i = 2.2$, and $\eta = 1.8$. The dotted line for $\sigma = 0.1$, the solid line for $\sigma = 0.2$, and the dotdashed line for $\sigma = 0.3$.

FIG. 7: Showing the mK-dV SWs profiles for $U_0 = 0.01$. $\Omega_c = 0.3$, $\sigma = 0.1$, $\delta = 60$, $Z_h = 1$, $H = 0.3$, and $\eta = 1.16$. The solid line for $\nu$ and $\nu - \Omega$, the dotted line for $\nu - \Omega - \Omega$ and $\nu$.
mK-dV equations, and analyzed their stationary SWs solutions based on some typical plasma parameters relevant to different astrophysical and laboratory plasma situations existed in some published works. We consider some typical plasma species density which is consistent with the relativistic degenerate astrophysical plasmas, e.g., \( n_i = 1 \times 10^{29} \text{ cm}^{-3} \), \( n_e = 9.1 \times 10^{29} \text{ cm}^{-3} \), \( n_{io} = 0.4 \times 10^{30} \text{ cm}^{-3} \), \((11, 36, 37, 48)\) and the other quantum parameters e.g., \( \beta = 0.1 \) to 0.9, \( H = 0.2 \) to 0.9 \((18, 22, 49)\), and ambient magnetic fields \( \sim 10^3 \text{G} \) \((15, 50)\). The values of these parameters may change depending on different plasma situations. The results, we have found from this investigation can be summarized as follows:

- The effect \( \sigma \) on \( V_p \) of the El acoustic waves for different values of \( \eta \) are displayed in Fig. 1. The phase speed of the acoustic waves decreases with the increase of \( \sigma \). This happens due to the increase of the inertia of the acoustic waves with the increase of the value of \( \sigma \). It is also observed that with the increase in \( \eta \), \( V_p \) of the El acoustic waves increases.
- The variation of width, \( \Delta_1 \) of the K-dV SWs with \( H \) is depicted in Fig. 2 for different values of \( \Omega_c \). It is found that as the values of both \( H \) and \( \Omega_c \) increase, \( \Delta_1 \) of the acoustic SWs decreases significantly.
- The variation of the K-dV SWs profiles with the \( \eta \) \((R)\) is shown in Fig. 3 (Fig. 4). It is observed that

our plasma system supports both positive (com-pressive) and negative (rarefactive) SWs structure. In case of \( \eta \), both amplitude and width of the SWs increases with the increase of \( \eta \). Increase of \( \eta \) means the increase of Fermi energy of the electrons and ions. But in case of \( R^i \), both amplitude and width of the SWs decreases with the increase of \( R^i \) \((\text{increase of relativistic effects of ions})\). Plasma particles can move freely in the weakly relativistic plasma more than the strong relativistic plasma.

4. The influences of the \( \Omega_c / \sigma \) on the mK-dV SWs profiles is shown in Fig. 5 (Fig. 6). It seems that there is no effect of \( \Omega_c \) on the amplitude of the mK-dV SWs but width decreases with the increase of \( \Omega_c \). It is also found that both width and amplitude of the mK-dV SWs decrease with the increase of \( \sigma \). This occurs due to the increase of the inertia of the plasma particles.

5. Fig. 7 compares the the amplitude and width of the quantum El SWs profiles for the relativistic case and the non-relativistic case. Plasma particles can move freely in the non-relativistic plasma more than the ultra-relativistic plasma due to the less number density of plasma species in non-relativistic case than the ultra relativistic case. Therefore, the amplitude as well as width of the SWs are noticeably higher for non-relativistic case than for ultra-relativistic case. The magnitude of the external magnetic field \( B_0 \) has no any effect on the amplitude of the SWs but it does have a direct effect on the width of SWs. We found that as the magnitude of \( B_0 \) \((\text{i.e.,} \Omega^R)\) increases, the width of SWs decreases, i.e., the magnetic field makes the solitary structures more spiky.

We hope that our lower order as well as higher order nonlinear analysis will be helpful for understanding the localized electrostatic disturbances not only in the different astrophysical degenerate compact objects \((\text{Shapiro}, 1983; \text{Garcia-Berro}, 2010; \text{Mamun}, 2010)\), but also in different dense laboratory plasma nonlinear experiments \((\text{Berezhiani, 1992; Murkhd, 2006})\).

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