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*International Journal of Current Research*
*Vol. 6, Issue, 04, pp.6180-6183, April, 2014*

**INTERNATIONAL JOURNAL OF CURRENT RESEARCH**

**ISSN: 0975-833X**

**RESEARCH ARTICLE**

# DIFFERENCE SETS IN ALGEBRAS

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**ARTICLE INFO**

***Article History:***
Received 16th January, 2013
Received in revised form
27th February, 2014
Accepted 20th March, 2014
Published online 23rd April, 2014

***Key words:***

Difference sets, Difference sets in algebra, Maximal difference set, Homomorphism of algebra, Banach algebra.

**ABSTRACT**

In this paper we introduce difference sets in algebra. We study their properties and prove some interesting results. We define a maximal difference set and show that a proper left difference set of an algebra with identity can be embedded in a maximal difference set. Then we prove difference set under homomorphism of one algebra to another. We also develop a difference set in context of a Banach algebra.



## I. Introduction

A ring is an additive abelian group R which is closed under a second operation called multiplication-the product of two elements x and y in R is written x y – in such a manner that

(i) multiplication is associative, that is if x, y, z are three elements in R, then x(y z) = (x y) z;

(ii) multiplication is distributive, i.e. if x, y, z are three elements in R, then

$$x ( y + z) = xy + xz$$
$$\text{and } (x + y) z = xz + yz.$$

R is called a commutative ring if x y = y x for all elements x and y in R. If the ring R contains a non-zero element 1 with the Since an algebra is also a ring, it may be commutative or non-commutative and may or may not have identity; and if it does have identity, then we can speak of its regular and singular elements. An algebra is real or complex according as the field

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property that x.1 = 1.x = x for any x, then 1 is called the identity elements and R is called a ring with identity. Let R be ring with identity. If x is an element in R, then it may happen that there is present in R an element y such that x y = y x = 1. In this case there is only one such element, and it is written as $x^{-1}$ and called the inverse of x. If an element x in R has an inverse then x is said to be regular. Elements which are not regular are called singular. Regular elements are often invertible elements, or non-singular elements.[1]

A linear space A is called an algebra if its vectors can be multiplied in such a way that A is also a ring in which scalar multiplication is related to multiplication by the following property:-

$$\alpha (x y) = (\alpha x) y = x (\alpha y)$$
$$\text{where } x , y \in A \text{ and } \alpha \text{ is a scalar.}$$

of scalars is the set of real or complex number respectively. A subalgebra of an algebra A is non-empty subset $A_0$ of A which is an algebra in its own right with respect to the operations in A. An ideal in an algebra A is defined to be a subset I with the following three properties:-

(i) I is a linear subspace of A;

(ii) $i \in I \Rightarrow x\, i \in I$ for every element $x \in A$;

(iii) $i \in I \Rightarrow i\, x \in I$ for every elements $x \in A$.[2]

We define a maximal left ideal in A to be a proper left ideal which is not properly contained in any other proper left ideal. We define the radical R of A to be the intersection of all its maximal left ideal. i.e. $R = \cap$ M L I.[3]

In analogy with these ideals we define a difference set in a real or complex algebra A to be subset G with the following three properties:-

(i) G is a difference set of A, regarding A as a linear space.

(ii) $g \in G \Rightarrow x\, g \in G$, for every element $x \in A$.

(iii) $g \in G \Rightarrow g\, x \in G$, for every element $x \in A$.

P be a partially ordered set with " $\leq$ " as partial ordering. An elements x in P is said to be maximal if $y \geq x \Rightarrow y = x$, i.e. if no element other than x itself is greater than or equal to x. Let A be a non-empty subset of a partially ordered set P. An element y in P is said to be an upper bound of A if $a \leq y$ for every $a \in A$. According to Zorn's lemma if P is a partially ordered set in which every chain has an upper bound then P possesses a maximal element.[4] A and A' be algebras which are both real or both complex. We define a homomorphism of A into A' to be a mapping f of A into A' which preserves all the operations in the sense that

$$f(x + y) = f(x) + f(y),$$

$$f(\alpha x) = \alpha f(x), \ \alpha \text{ being any scalar,}$$

$$\text{and } \ f(x y) = f(x) f(y).$$

An isomorphism is one-one homomorphism and A is said to be isomorphic to A' if there exists an isomorphism of A onto A'.[5]

A Banach algebra is a complex Banach space which is also an algebra with identity 1 and in which the multiplicative structure is related to the norm by the following requirements:-

(i) $\| x y \| \leq \| x \| \, . \, \| y \|$,
(ii) $\| 1 \| = 1$.

It follows that $x_n \to x, y_n \to y \Rightarrow x_n y_n \to xy$.[1]

Suppose T is a topology on a vector space X such that

a) every point of X is a closed set, and
b) the vector space operations are continuous with respect to T, then T is said to be a vector topology on X, and X is called a topological vector space.

The closure $\bar{E}$ of $E \subseteq X$ is the intersection of all closed sets that contain E.[6]

## II. Theorems

**Theorem1:** Let A be an algebra. Let $G_1, G_2, \ldots, G_n$ be difference sets of A and $\alpha_1, \alpha_2, \ldots, \alpha_n$ be scalars then $\sum_{i=1}^{n} \sim_i G_i$ is a difference set of A.

Proof : $\sum_{i=1}^{n} \sim_i G_i$ is a difference set regarding A as a linear space.

Let $z \in \alpha_1 G_1 + \alpha_2 G_2 + \ldots + \alpha_n G_n$

We can write

$$z = \alpha_1 g_1 + \alpha_2 g_2 + \ldots + \alpha_n g_n,$$

where $g_i \in G_i$ , $i = 1,2,\ldots,n$.

let $x \in A$, then

$$x z = x(\alpha_1 g_1 + \alpha_2 g_2 + \ldots + \alpha_n g_n)$$

$$= \alpha_1 x g_1 + \alpha_2 x g_2 + \ldots + \alpha_n x g_n$$

$\in \alpha_1 G_1 + \alpha_2 G_2 + \ldots + \alpha_n G_n$ (Since $G_i$ is a difference set of algebra A therefore $xg_i \in G_i$ )

and

$$z x = (\alpha_1 g_1 + \alpha_2 g_2 + \ldots + \alpha_n g_n) x$$

$$= \alpha_1 g_1 x + \alpha_2 g_2 x + \ldots + \alpha_n g_n x$$

$\in \alpha_1 G_1 + \alpha_2 G_2 + \ldots + \alpha_n G_n$ . Hence $\alpha_1 G_1 + \alpha_2 G_2 + \ldots + \alpha_n G_n$ is a difference set of A.

**Theorem 2 :** Let G be a left difference set of an algebra A with identity 1. If $1 \in G$ then $G = A$.

Proof: Since G is a left difference set of A, then

$$G \subseteq A \qquad (1)$$

Let $x \in A$ then since $1 \in G$, $x.1 \in G$, or $x \in G$.

Thus $x \in A \Rightarrow x \in G$.

Hence $A \subseteq G(2)$ (2)

from (1) and (2) it follows that $G = A$.

Similarly if G is a right difference set of an algebra A with identity 1 such that $1 \in G$ then $G = A$.

Finally, if G is a difference set of an algebra A with identity 1 such that $1 \in G$ then $G = A$.

**Theorem3:** Let G be a proper left difference set of an algebra A with identity 1. G can be embedded in a maximal left difference set of A.

Proof: Let P be a partially ordered set of all proper left difference sets of A containing G, partially ordered by set inclusion.
Let $\{G_i\}$ be a chain in P, i.e., it is a totally ordered family of proper difference sets of A each containing G.

Since $G \in \{G_i\}$ this family is non-empty.

Let $H = U_i\, G_i$ , H is a difference set of A regarding A as a linear space.

Let g ∈ H, then g ∈ $G_i$ for some i . since $G_i$ is a left difference set,

$$x \in A \Rightarrow x g \in G_i$$

$$\Rightarrow x g \in H.$$

Therefore H is a left difference set containing G. Since Gi is a proper left difference set then by theorem2,

$$1 \notin G_i .$$

Hence $1 \notin H$. Thus H is a proper left difference set of A containing G.

Therefore H ∈ {$G_i$}.

Also for any i, $G_i \subseteq H$.

Thus H is an upper bound of {$G_i$}.

Since {$G_i$} is any chain in P, we see that every chain in P has an upper bound. Hence by Zorn's lemma, if G is not itself a maximal left difference set then there exists a maximal left difference set G' of A such that $G \subseteq G' \subseteq A$.

Thus G can be embedded in a maximal left difference set of A.

Thus any proper left difference set in A can be embedded in a maximal left difference set of A. Since {0} is a proper left difference set, maximal left difference sets certainly exist.

**Theorem4:** Let A be an algebra with identity 1. Let G be a left difference set of A such that G contains a left regular elements then A = G.

Proof: Let G contain a left regular element x then there exists another element y such that y x =1.

Since x ∈ G, y x ∈ G.

Hence 1 ∈ G.

Therefore by theorem2, G = A.

Similarly, if G is a right difference set containing a right element the G = A.

Finally, if G is a difference set containing a regular element then G = A.

Thus any proper difference set of A cannot contain a regular element.

**Theorem5:** Let A , A' be algebras with the same field of scalars. Let f be a homomorphism of A onto A' , then the image of each difference set in A is a difference set in A' and the inverse image of a difference set in A' is a difference set in A.

Proof: Let G be a difference set of A. Since f is also a linear transformation of linear space A onto A', then f(G) is a difference set of linear space A'.

Let z ∈ f (G) then z = f(g) for g ∈ G.

Let x ∈ A'. Since f is onto, there exists a ∈ A such that x = f(a).

Therefore x z = x f(g) = f(a) f(g) = f(a g) ∈ f(G), for a g ∈ G.

Also z x = f(g) f(a) = f(g a) ∈ f(G), for g a ∈ G.

Hence f(G) is a difference set of A'.

Let H be a difference set of A'. Then by theorem2, $f^{-1}(H)$ is a difference set of linear space A.

Let x ∈ $f^{-1}(H)$, then f(x) ∈ H.

Let a ∈ A, then f(a) ∈ A'.

Hence f(a x) = f(a) f(x) ∈ H

$$\Rightarrow a x \in f^{-1}(H).$$

Also f( x a ) = f(x) f(a) ∈ H

$$\Rightarrow x a \in f^{-1}(H).$$

Hence $f^{-1}(H)$ is a difference set in A.

**Theorem6:** Let A be a Banach algebra and G a proper left difference set of A, then $\overline{G}$ is also a proper left difference set of A.

First we prove **Lemma**: If A is difference set of a topological vector space X then Ā is also a difference set.

Proof: Let x , y ∈ Ā then

$$x - y \in \bar{A} - \bar{A}.$$

Now $\overline{\sim A} + \overline{A} \subseteq \overline{\sim A + A}$.[3]

Putting α = 1 and β = -1, we have

$$\bar{A} - \bar{A} \subseteq \overline{A - A}.$$

Hence $x - y \in \bar{A} - \bar{A} \subseteq \overline{A - A}$.

Since A is a difference set,

$$A - A \subseteq A \Rightarrow \overline{A - A} \subseteq \bar{A}.$$

Therefore $x - y \in \bar{A}$.

This shows that Ā is a difference set.

**Proof of the Theorem6:** Since G is a difference set of topological vector space A, then by above lemma, $\overline{G}$ is also a difference set of linear space A.

Let g ∈ $\overline{G}$, then there exists a sequence $\{g_n\} \subseteq G$ such that $g_n \to g$.

Let x ∈ A, then $x g_n \to x g$. But $\{x g_n\} \subseteq G$ and hence $x g \in \overline{G}$.

Therefore $\overline{G}$ is a left difference set of A.

Since G is a proper left difference set by theorem4, it cannot contain a regular element.

Let S denote the set of singular elements of A,

then $G \subseteq S$.

Now S is a closed set.

Thus $G \subseteq \overline{G} \subseteq S$.

Since $1 \notin S$, $1 \notin \overline{G}$.

Hence $\overline{G}$ is a proper left difference set of A. This completes the proof.

Similarly, if G is a proper right difference set of A, then $\overline{G}$ is also a proper right difference set of A.

Finally, if G is a proper difference set of A then $\overline{G}$ is a proper difference set of A.

### III. Concluding Remarks

Thus we have discussed difference sets, difference sets in algebra, homomorphism of difference set in algebra and difference set in a Banach algebra and proved interesting results.

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