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RESEARCH ARTICLE

# k-HERMITIAN DOUBLY STOCHASTIC, s- HERMITIAN DOUBLY STOCHASTIC AND s-k- HERMITIAN DOUBLY STOCHASTIC MATRICES

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ABSTRACT

The basic concepts and theorems of k-Hermitian doubly stochastic, s- Hermitian doubly stochastic and s-k- Hermitian doubly stochastic matrices are introduced with examples.



## INTRODUCTION

We have already seen the concept of Hermitian doubly stochastic matrices. In this paper introduce the Hermitian doubly stochastic matrix is developed in complex matrices. Recently Hill and Waters (1992) have developed a theory of k-real and k-Hermitian matrices as a generalization of s-real and s-Hermitian matrices. Ann Lee (1976) has initiated the study of secondary Hermitian matrices, that is matrices whose entries are symmetric about the secondary diagonal. Ann Lee (1976) has shown that the matrix A, the usual conjugate A and secondary conjugate of A are related as $\bar{A}$ = VA*V and A* = V $\bar{A}$ V where V is a permutation matrix with units in the secondary diagonal.

### DEFINITION: 1 (2014)

A matrix A $\in C^{n \times n}$ is said to be Hermitian doubly stochastic matrix if A = A*and
$\sum_{i=1}^{n}|a_{ij}|$ = 1, j = 1, 2, ………n
and $\sum_{j=1}^{n}|a_{ij}|$ = 1, i = 1, 2, ………n and all $|a_{ij}| \geq 0$.
If A is doubly stochastic and also Hermitian then it is called a Hermitian doubly stochastic matrix.

### DEFINITION: 2

A matrix A $\in C^{n \times n}$ is said to be k- Hermitian doubly stochastic matrix if $\bar{A}$ = K A* K
Where K is a permutation matrix and K = (1) (2 3).

### LEMMA:

For A is k-Hermitian doubly stochastic matrix then the following are equivalent.
(i) $\bar{A}$ = KA*K and A*=K $\bar{A}$ K (ii) KA* = KA (iii) A*K = AK (iv) (KA)* = A*K (v) (A*K)* = KA

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**EXAMPLE:**

A = $\begin{pmatrix} 1 & -i & i \\ i & 1 & -i \\ -i & i & 1 \end{pmatrix}$ $\bar{A}$ = $\begin{pmatrix} 1 & i & -i \\ -i & 1 & i \\ i & -i & 1 \end{pmatrix}$ A* = $\begin{pmatrix} 1 & -i & i \\ i & 1 & -i \\ -i & i & 1 \end{pmatrix}$ and k = (1) (2 3) = $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

(i) K A*K = $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -i & i \\ i & 1 & -i \\ -i & i & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -i & i \\ i & 1 & -i \\ -i & i & 1 \end{pmatrix} = \bar{A}$

K $\bar{A}$ K = $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & i & -i \\ -i & 1 & i \\ i & -i & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -i & i \\ i & 1 & -i \\ -i & i & 1 \end{pmatrix}$ = A*

(ii) KA* = $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -i & i \\ i & 1 & -i \\ -i & i & 1 \end{pmatrix} = \begin{pmatrix} 1 & -i & i \\ -i & i & 1 \\ i & 1 & -i \end{pmatrix}$ = KA

(iii) A*K = $\begin{pmatrix} 1 & -i & i \\ i & 1 & -i \\ -i & i & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & i & -i \\ i & -i & 1 \\ -i & 1 & i \end{pmatrix}$ = AK

(iv) (KA)* = $\begin{pmatrix} 1 & i & -i \\ i & -i & 1 \\ -i & 1 & i \end{pmatrix}$ = A*K(vi) (A*K)* = $\begin{pmatrix} 1 & -i & i \\ -i & i & 1 \\ i & 1 & -i \end{pmatrix}$ =KA

(v)

**RESULTS:** KA = $\overline{AK}$ and AK = $\overline{KA}$

**THEOREM: 1**

Let A ∈ $C^{n \times n}$ is k-Hermitian doubly stochastic matrix then $\bar{A}$ = K A* K.

**Proof:**

K A* K = KAK where KA* = KA
= $\overline{AK}$K where KA = $\overline{AK}$
= $\bar{A}\bar{K}$ K = $\bar{A}K^2$ where $\bar{K}$= K
= $\bar{A}$ where $K^2$ = I

**THEOREM: 2**

Let A ∈ $C^{n \times n}$ is k-Hermitian doubly stochastic matrix then K$\bar{A}$ K = A*.

**Proof:**

K $\bar{A}$ K = K$\bar{A}\bar{K}$where K = $\bar{K}$
= K$\overline{AK}$= KKA where $\overline{AK}$ = KA
= KKA* where KA = KA*
= $K^2$A* = A* where $K^2$ = I

**THEOREM: 3**

Let A, B ∈ $C^{n \times n}$ is k-Hermitian doubly stochastic matrix then $\frac{1}{2}$(A + B ) is k-Hermitian doubly stochastic matrix.

**Proof:** Let A and B are k-Hermitian doubly stochastic matrix if $\bar{A}$ = K A* K and $\bar{B}$ = K B* K .
To prove $\frac{1}{2}$ (A + B) is k-Hermitian doubly stochastic matrix we will show that $\frac{1}{2}$ ($\overline{A+B}$)=K $\frac{1}{2}$ (A + B)* K
Now K $\frac{1}{2}$(A + B )* K = K $\frac{1}{2}$(A* + B* ) K = $\frac{1}{2}$K (A* + B *) K = $\frac{1}{2}$ (K A* + K B *) K
= $\frac{1}{2}$ (K A* K+ K B* K) = $\frac{1}{2}$ ($\bar{A}+\bar{B}$ ) = $\frac{1}{2}$ ($\overline{A+B}$) where $\bar{A}$ = K A* K and $\bar{B}$ = K B* K.

**THEOREM: 4**

If A and B are k-Hermitian doubly stochastic matrix then AB is also k-Hermitian doubly stochastic matrix.

**Proof:** Let A and B are k-Hermitian doubly stochastic matrix if $\bar{A}$ = K A* K and $\bar{B}$ = K B* K

Since A*and B*are also k-Hermitian doubly stochastic matrices then A* = K $\bar{A}$ K and B* = K$\bar{B}$ K.
To prove A B is k-Hermitian doubly stochastic matrix we will show that AB = $\overline{BA}$ = K (A B )* K
Now K (A B )* K= K(B*A*)K

= K(K $\bar{B}$ K)( K $\bar{A}$ K)Kwhere A* = K $\bar{A}$ K and B* = K$\bar{B}$ K.
= $K^2\bar{B}$ $K^2\bar{A}$ $K^2$ = $\bar{B}$ $\bar{A}$ where $K^2$ = I
= $\overline{BA}$ = AB

**DEFINITION: 3**

A matrix A ∈ $C^{n \times n}$ is said to be s-Hermitian doubly stochastic matrix if $\bar{A}$= VA*V
Where V is a exchange matrix.

**LEMMA:**

For A is s-Hermitian doubly stochastic matrix then the following are equivalent.
(i) $\bar{A}$= V A* V and A* = V $\bar{A}$V (ii)VA* = VA (iii) A*V = AV (iv) (VA)* = A*V (v) (A*V)* = VA

**EXAMPLE:**

$$A = \begin{pmatrix} 1 & -i & i \\ i & 1 & -i \\ -i & i & 1 \end{pmatrix} \quad \bar{A} = \begin{pmatrix} 1 & i & -i \\ -i & 1 & i \\ i & -i & 1 \end{pmatrix} \quad A^* = \begin{pmatrix} 1 & -i & i \\ i & 1 & -i \\ -i & i & 1 \end{pmatrix} \quad V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

(i) $$V A^* V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -i & i \\ i & 1 & -i \\ -i & i & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & i & -i \\ -i & 1 & i \\ i & -i & 1 \end{pmatrix} = \bar{A}$$

$$V\bar{A} V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & i & -i \\ -i & 1 & i \\ i & -i & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & i & -i \\ -i & 1 & i \\ i & -i & 1 \end{pmatrix} = A^*$$

(ii) $$VA^* = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -i & i \\ i & 1 & -i \\ -i & i & 1 \end{pmatrix} = \begin{pmatrix} -i & i & 1 \\ i & 1 & -i \\ 1 & -i & i \end{pmatrix} = VA$$

(iii) $$A^*V = \begin{pmatrix} 1 & -i & i \\ i & 1 & -i \\ -i & i & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} i & -i & 1 \\ -i & 1 & i \\ 1 & i & -i \end{pmatrix} = AV$$

(iv) $$(VA)^* = \begin{pmatrix} i & -i & 1 \\ -i & 1 & i \\ 1 & i & -i \end{pmatrix} = A^*V$$ and (v) $$(A^*V)^* = \begin{pmatrix} -i & i & 1 \\ i & 1 & -i \\ 1 & -i & i \end{pmatrix} = VA$$

**RESULTS:** VA = $\overline{AV}$ and AV = $\overline{VA}$

**THEOREM: 5**

Let A ∈ $C^{n \times n}$ is s-Hermitian doubly stochastic matrix then $\bar{A}$ = V A* V.

**Proof:**

V A* V= VAV where VA* = VA
= $\overline{AV}$ V where VA = $\overline{AV}$
= $\bar{A}$ $\bar{V}$V = $\bar{A}$ $V^2$ where $\bar{K}$ = K
= $\bar{A}$ where $V^2$ = I

**THEOREM: 6**

Let A ∈ $C^{n \times n}$ is s-Hermitian doubly stochastic matrix then A* = V $\bar{A}$V.

**Proof**

V $\bar{A}$ V = V$\bar{A}$ $\bar{V}$ where V = $\bar{V}$
= V$\overline{AV}$ = VVA where $\overline{AV}$ = VA
= VVA* where VA = VA*
= $V^2$A* = A* where $V^2$ = I

**THEOREM: 7**

Let A, B $\in C^{n \times n}$ is s-Hermitian doubly stochastic matrix then $\frac{1}{2}$(A + B ) is s-Hermitian doubly stochastic matrix.

**Proof:**

Let A and B are s-Hermitian doubly stochastic matrix if $\bar{A}$ = V A* V and $\bar{B}$= V B* V .
To prove $\frac{1}{2}$(A + B) is s-Hermitian doubly stochastic matrix we will show that $\frac{1}{2}(\overline{A+B})$=V $\frac{1}{2}$(A + B)* V
Now V $\frac{1}{2}$(A + B )* V = V $\frac{1}{2}$(A* + B* ) V = $\frac{1}{2}$V (A* + B *) V = $\frac{1}{2}$(V A* + V B *) V
= $\frac{1}{2}$(V A* V+ V B* V) = $\frac{1}{2}(\bar{A}+\bar{B})$ = $\frac{1}{2}(\overline{A+B})$ where $\bar{A}$ = V A* V and $\bar{B}$ = V B* V

**THEOREM: 8**

If A and B are s-Hermitian doubly stochastic matrix then AB is also s-Hermitian doubly stochastic matrix.

**Proof:** Let A and B are s-Hermitian doubly stochastic matrix if $\bar{A}$ = V A* V and $\bar{B}$ = V B* V .
Since A*and B*are also s-Hermitian doubly stochastic matrices then A* = V $\bar{A}$ V and B* = V $\bar{B}$ V
To prove A B is s-Hermitian doubly stochastic matrix we will show that AB = $\overline{BA}$ = V (A B )* V
Now V (A B )* V= V(B*A*)V = V(V $\bar{B}$ V)( V $\bar{A}$ V)V where A* = V $\bar{A}$ V and B* = V $\bar{B}$ V
= $V^2\bar{B}\ V^2\bar{A}\ V^2$ = $\bar{B}\ \bar{A}$ where $V^2 = I$
= $\overline{BA}$ = AB

**DEFINITION: 4 (2009)**
A matrix A $\in C^{n \times n}$ is said to be s-k-Hermitian doubly stochastic matrix if

(i) A = KVA*VK (ii) $\bar{A}$ = KV$\bar{A}$ VK
(iii) A = VKA*KV (iv) $\bar{A}$ = VK$\bar{A}$ KV

Where V is a exchange matrix and K is a permutation matrix and K = (1) (2 3).

**THEOREM: 9**

Let A $\in C^{n \times n}$ is s-k-Hermitian doubly stochastic matrix then

(i) A* = KVA*VK (ii) $\bar{A}$ = KV$\bar{A}$ VK
(iii) A* = VKA*KV (iv) $\bar{A}$ = VK$\bar{A}$ KV

**Proof:**

KVA*VK = K(VA*V)K = K$\bar{A}$ K where VA*V = $\bar{A}$
= A* where K$\bar{A}$ K = A*
KV$\bar{A}$ VK = K(V$\bar{A}$ V)K = KA*K where V$\bar{A}$ V = A*
= $\bar{A}$ where KA*K = $\bar{A}$
VKA*KV = V(KA*K)V = V$\bar{A}$V where KA*K = $\bar{A}$
= A* where V$\bar{A}$V = A*
VK$\bar{A}$KV =V(K$\bar{A}$K)V = V A* V where K$\bar{A}$K = A*
= $\bar{A}$ where V A* V = $\bar{A}$

**EXAMPLE:**

$$A = \begin{pmatrix} 1 & -i & i \\ i & 1 & -i \\ -i & i & 1 \end{pmatrix} \quad \bar{A} = \begin{pmatrix} 1 & i & -i \\ -i & 1 & i \\ i & -i & 1 \end{pmatrix} A^* = \begin{pmatrix} 1 & -i & i \\ i & 1 & -i \\ -i & i & 1 \end{pmatrix}$$

$$K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad KV = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad VK = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

(i) $$KVA^*VK = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -i & i \\ i & 1 & -i \\ -i & i & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -i & i \\ i & 1 & -i \\ -i & i & 1 \end{pmatrix} = A^*$$

(ii) $KV\bar{A}VK = \begin{pmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}1 & i & -i\\ -i & 1 & i\\ i & -i & 1\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix} = \begin{pmatrix}1 & i & -i\\ -i & 1 & i\\ i & -i & 1\end{pmatrix} = \bar{A}$

(iii) $VKA^*KV = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & -i & i\\ i & 1 & -i\\ -i & i & 1\end{pmatrix}\begin{pmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix} = \begin{pmatrix}1 & -i & i\\ i & 1 & -i\\ -i & i & 1\end{pmatrix} = A^*$

(iv) $VK\bar{A}KV = = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & i & -i\\ -i & 1 & i\\ i & -i & 1\end{pmatrix}\begin{pmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix} = \begin{pmatrix}1 & i & -i\\ -i & 1 & i\\ i & -i & 1\end{pmatrix} = \bar{A}$

**THEOREM:10**

Let A, B $\in R^{n \times n}$ is s-k-Hermitian doubly stochastic matrix then $\frac{1}{2}(A + B)$ is s-k-Hermitian doubly stochastic matrix.

**Proof:** Let A and B are s-k-Hermitian doubly stochastic matrix if $A^* = KV A^* VK$ and $B^* = KVB^*VK$. To prove $\frac{1}{2}(A+B)$ is s-k-Hermitian doubly stochastic matrix we will show that $\frac{1}{2}(A+B)^* = KV \frac{1}{2}(A+B)^*VK$

Now $KV \frac{1}{2}(A + B)^* VK = K( V \frac{1}{2}(A + B)^* V)K = K \frac{1}{2} (\overline{A + B}) K$ using theorem (7)

$= \frac{1}{2}(A + B)^*$ using theorem (3)

**THEOREM: 11**

If A and B are s-k-Hermitian doubly stochastic matrix then AB is also s-k-Hermitian doubly stochastic matrix.

**Proof:**

Let A and B are s-k-Hermitian doubly stochastic matrix if $A^* = KV A^* VK$ and $B^* = KVB^*VK$.
To prove A B is s-k-Hermitian doubly stochastic matrix we will show that $(AB)^* = KV(A B)^* VK$
Now $KV(A B)^* VK = K(V(A B)^* )VK = K(\overline{BA}) K$ using theorem (8)
$= (AB)^*$ using theorem (4)

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