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RESEARCH ARTICLE

RANK OF MAXIMAL SUBGROUP OF A FULL TRANSFORMATION SEMIGROUP

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ABSTRACT

We shall show that the rank of maximal subgroup of a full transformation semigroup on a finite set equal to the lower bound for the rank $r_n(T_n)$

INTRODUCTION

In this section, we defined terms and gives some examples relating to this work. And in section two (2) we show that the rank of maximal subgroup of a full transformation semigroup on a finite set equal to the lower bound for the rank $R_n(T_n)$

Definition (Full transformation semigroup)

Let X be a non-empty set, A full transformation semigroup (T_X, o) consist of T_X (maps α: X→X), and a composition o

Remark

If α and β are maps from X into X, then

$x(\alpha o\beta)$ = x $(\alpha\beta)$ = (xα)β $(x\in X)$

When X is a finite set, we shall write T_n in place of T_X, where n = $|X|$, the order of the set X. we shall usually consider the base set of T_n to be $X_n = \{1,2,\dots,n\}$, and a typical member $\alpha \in T$ can then be specified by listing the images of the members of X_n in order $(1\alpha, 2\alpha, \dots n\alpha)$. We shall denote the rank r of $\alpha \epsilon T_X$ as $|rank\ \alpha|$.

Note

The number of elements in symmetric semigroup (G_X, o) of all permutations of a set X is given as

$|G_X|$=n!, $|T_X| = n^n$

Where n=$|X|$

G_X Consisting of all bijections from X onto X,
G_X is a subsemigroup of T_X

Definition (Maximal Subgroup)

Maximal subgroup of a semigroup S is a subgroup (that is, a subsemigroup which forms a group under the semigroup operation of S) which is not properly contained in another subgroup of S.

Example

For any n, the alternating group A_n of degree n is maximal in the symmetric group S_n of degree n, ie

$|A_n| = \frac{n!}{2}, n > 1$ and $|S_n|$ = n!

Definition

Let S be a semigroup. We say that an element c of S is a group element if and only if c falls in some maximal subgroup of S. Otherwise c is a non-group element. Let NG be a set which consist of all nongroup elements of S. if every element of S is expressible as a product of elements of NG, then the non-group rank of S is defined by

Ng rank S = min $\{|A| : A\subseteq NG, \langle A\rangle = S\}$

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Example

The dihedral subgroup dih_4; the 5 – element layer form the non-group element since the element in that layer can be express as a product of element of NG.
The non-group rank of the Dih_4 is

NG rank $Dih_4 = \min\{|e| : e \subseteq NG, \langle e\rangle = Dih_4\} = 5$

Definition (Stirling Number)

The stirling number of the second kind SCn,r is the number of ways to partition a set of n objects into r non-empty subjects and can also be denoted by $\{{n \atop r}\}$.

This occurs in the field of mathematics called combinatories and the study of partitions.

This can be calculated using the following explicit formular

$$\left\{{n \atop r}\right\} = \frac{1}{r!}\sum_{i=0}^{r}(-1)^i \binom{r}{i}(r-i)^n \qquad \ldots\ldots\ldots\ldots \quad (1)$$

The symbol $\left\{{n \atop r}\right\}$ is read as "n subset r", r is called the weight or the cardinality.

From Howie classification of rank (r_1, r_2, r_3, r_4, r_5).
We observe that r_2 (S) is what is normally called rank, which has been extensively studied.

r_2 (S) = min{k: there exists a subset U of S cardinality K such that U generates S}.

We notice that our definition of maximum subgroup coincide with the way he defined r4(S) (the upper rank). We now find the lower bound of r_4(S) simply by producing an independent set in S.

r_4 (S) =max {k: ∃ a subset U of S cardinality K which is independent}.

Definition

Let $\alpha \in T_n$ and let $x_0 \in X_n$ be any element. If $x\alpha = x_0$ for every element $x \in X_n$, then α is called a constant transformation. The image, Defect set, defect and kernel of α are define by

im $(\alpha) = \{x\alpha : x \in X_n\}$
Def $(\alpha) = X_n \setminus$ im (α)
def $(\alpha) = |$ Def $(\alpha)|$
Ker $(\alpha) = \{(x,y) \in X_n \times X_n : x\alpha = y\alpha\}$

respectively. Now we state the lemma which will be useful in this work.

Lemma

For any $x, \beta \in T_n$
Ker $(\alpha) \subseteq$ Ker $(\alpha\beta)$
im $(\alpha\beta) \subseteq$ im (β)

Proof

From Green's relations, α, β are $\mathcal{R}$-equivalent if and only if ker α = Ker β – where for a given in $\emptyset$ $sing_n$, the equivalence relation ker $\emptyset$ (=$\emptyset o \emptyset^{-1}$) is defined as $\{(x,y) \in X \times X : x\emptyset = y\emptyset\}$ Suppose now that α,β in J_{n-1} are such the $\alpha\beta$ also lies in J_{n-1}. Then ker $\alpha\beta \supseteq$ ker α,

While

$|X/\ker\alpha\beta| = |X/\ker\alpha|$ =n-1
It follows that ker $\alpha\beta$ =ker α ie. That $\alpha\beta\mathcal{R}\alpha$
(ii)The same argument follows and we have that $\alpha\beta\mathcal{L}\beta$ and that im $(\alpha\beta) \subseteq$ im (β)

If α, β are $\mathcal{L}$-equivalent.

The idea behind Green's equivalences is to sort out the elements of a semigroup

Each D-class in a semigroup S is a union of $\mathcal{L}$-classes and $\mathcal{R}$-classes. The intersection of an $\mathcal{L}$-class and $\mathcal{R}$-class is either empty or is an H-class. Hence it is convenient to visualize a D-class as eggbox, in which each row represents an $\mathcal{R}$-class, each column represents an $\mathcal{L}$-class and each cell represents an H-class. (It is possible for the eggbox to contain a single row or a single column of cells, or even to contain only one cell.) One can then analysis a semigroup, by finding these uniform blocks and describing connections between them.

$(\alpha,\beta) \in \mathcal{L} \Leftrightarrow$ im (α) = im (β)
$(\alpha,\beta) \in \mathcal{R} \Leftrightarrow$ ker (α) = ker (β)
$(\alpha,\beta) \in H \Leftrightarrow$ im (α) = im (β) and ker (α) = ker (β)
$(\alpha,\beta) \in D \Leftrightarrow |$im $(\alpha)| = |$im $(\beta)|$

$\alpha \in T_n$ is an idempotent element if and only if the restriction of α to im (α) is the identity map on im (α). If e is an idempotent in a semigroup S, then H_e is a subgroup of S. No H-class in S can contain more than one idempotent. We denote the D-Green class of all self maps of defect r by D_{n-r} (o≤ r ≤ n-1).

Let D_k be a D-class then
D_k has $\binom{n}{k}$ distinct $\mathcal{L}$-classes
D_k has $S(n,k)$ distinct $\mathcal{R}$-Classes
D_k has $S(n,k)\binom{n}{k}$ distinct H-classes
Each H-class has $k!$ elements
Each $\mathcal{L}$-Class has k^{n-k} group H-classes.
Here $S(n,k)$, for $1 < k \leq n$ is as defined by (1)
Where $k = r$ and $i = r$ in (1)

Which satisfies recurrence relations below

$S(1,1) = S(n,n) = 1$ and $S(n+1,k) = S(n,k-1) + kS(n,k)$.

The Rank r_4 of T_n

We compute here the lower bound of $r_4(T_n)$ which is defined as our maximal rank subgroup of T_n.

Theorem

$r_4(T_n) \geq |A| = [S(n, n-1) - (n-1)](n-1)! + n + 1.$

Proof

We start with the construction of an independent subset of T_n. For $X_n = \{1, \ldots, n\}$ and $Y = X_n \backslash \{i\}, i = 1, \ldots, n$ we define a set denoted by A. consider the $\mathcal{L}$-class L_Y in D_{n-1}. The set A contains only idempotent of group H-classes in L_Y and all elements of non-group H-classes in L_Y, and also α, is the contants transformation such as $x\alpha = i$ for all $x \in X_n$ and β, is the identity map of T_n.

The Cardinality of A is

$$[S(n, n-1) - (n-1)](n-1)! + n + 1$$

Here $S(n, n-1)$ is the number of $\mathcal{R}$-classes in Dn-1, (n-1) is the number of group H-classes and (n-1)! is the cardinality of each of the H-classes.

A is an independent subset of T_n. Since

$\alpha = \begin{pmatrix} 1\,2 \cdots n \\ i\,i \cdots i \end{pmatrix}$ and $\beta = \begin{pmatrix} 1\,2 \cdots n \\ 1\,2 \cdots n \end{pmatrix}$

Constant transformation and permutation respectively, From lemma 1.10 $\alpha \notin \langle A \backslash \{\alpha\} \rangle$. And $\beta \notin \langle A \backslash \{\beta\} \rangle$. Without loss of generality we may assume that $A = A \backslash \{\alpha, \beta\}$. For any $\gamma \in A$, this is enough to show that $\gamma \notin \langle A \backslash \{\gamma\} \rangle$. Now suppose that $\gamma \in \langle A \backslash \{\gamma\} \rangle$. Then there exist $\delta_1, \delta_2, \ldots, \delta_k \in A \backslash \{\gamma\}$ such that $\delta_1, \delta_2, \ldots, \delta_k = \gamma$.

Since $\delta_1, \delta_2, \ldots, \delta_k, \gamma \in D_{n-1}$. We find from lemma 1.10.

Ker (δ_1) = ker (γ) and im (γ) = im (δ_k). ………….. (2)

Since $\delta_1, \delta_2, \ldots, \delta_k, \gamma \in L_Y$ then

Im (γ) = im (δ_j), j=1,2,…,k. ………….. (3)

From equation (2) and (3), ker (γ) = Ker (δ_1) and im (γ) = im (δ_1). That is, γ and δ_1 are in the same H-class.

If γ is an idempotent, δ_1 is not in the H_γ-class, Because this contradicts with the definition of A. So there is no element in $A \backslash \{\gamma\}$ such that ker (δ_1) = ker (γ). Hence $\gamma \notin \langle A \backslash \{\gamma\} \rangle$

If γ is not an idempotent, since each non-group H-class is not closure (indeed is an idempotent subset) and idempotents of group H-classes are right identity of the same $\mathcal{L}$-classes then $|\text{im}\,\gamma| = |\text{im}(\delta_1\delta_2 \ldots \delta_k)| \leq n-2$.

Since $\gamma \in D_{n-1}$, this is in contradiction with $|\text{im}\,\gamma| = n-1$. So $\gamma \notin \langle A \backslash \{\gamma\} \rangle$.

Example

For n=3, T_3 has 3 D −Green classes like below:

D_3

$\begin{pmatrix}\mathbf{1} & \mathbf{2} & \mathbf{3}\\ \mathbf{1} & \mathbf{2} & \mathbf{3}\end{pmatrix}$	$\begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{pmatrix}$	$\begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\end{pmatrix}$	$\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}$	$\left(\begin{matrix}1 & 2\\ 3 & 1\end{matrix}\right.$

D_2

$\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 2\end{pmatrix}$ $\left(\begin{matrix}1 & 2\\ 2 & 1\end{matrix}\right.$	$\begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 3\end{pmatrix}$ $\left(\begin{matrix}1 & 2\\ 3 & 1\end{matrix}\right.$	$\begin{pmatrix}\mathbf{1} & \mathbf{2} & \mathbf{3}\\ \mathbf{2} & \mathbf{3} & \mathbf{3}\end{pmatrix}$ $\left(\begin{matrix}\mathbf{1} & \mathbf{2}\\ \mathbf{3} & \mathbf{2}\end{matrix}\right.$
$\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 1\end{pmatrix}$ $\left(\begin{matrix}1 & 2\\ 2 & 1\end{matrix}\right.$	$\begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 1\end{pmatrix}$ $\left(\begin{matrix}1 & 2\\ 3 & 1\end{matrix}\right.$	$\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 2\end{pmatrix}$ $\left(\begin{matrix}\mathbf{1} & \mathbf{2} & \mathbf{3}\\ \mathbf{3} & \mathbf{2} & \mathbf{3}\end{matrix}\right.$
$\begin{pmatrix}1 & 2 & 3\\ 1 & 1 & 2\end{pmatrix}$ $\left(\begin{matrix}1 & 2\\ 2 & 2\end{matrix}\right.$	$\begin{pmatrix}1 & 2 & 3\\ 1 & 1 & 3\end{pmatrix}$ $\left(\begin{matrix}1 & 2\\ 3 & 3\end{matrix}\right.$	$\begin{pmatrix}\mathbf{1} & \mathbf{2} & \mathbf{3}\\ \mathbf{2} & \mathbf{2} & \mathbf{3}\end{pmatrix}$ $\left(\begin{matrix}1 & 2\\ 3 & 3\end{matrix}\right.$

D_1

$\begin{pmatrix}\mathbf{1} & \mathbf{2} & \mathbf{3}\\ \mathbf{1} & \mathbf{1} & \mathbf{1}\end{pmatrix}$	$\begin{pmatrix}1 & 2 & 3\\ 2 & 2 & 2\end{pmatrix}$	$\begin{pmatrix}1 & 2 & 3\\ 3 & 3 & 3\end{pmatrix}$

$X_3 = \{1, 2, 3\}$ and let i=1. Since Y ={2, 3}, consider the $L_Y = L_{\{2,3\}}$ class in top D − class D_2. For A, we take idempotents of the group $H - class$es and all elements of non-group$H - class$es in this L-class. Also α is constant transformation in D_1 and β is identity map in D_3. Briefly the set A consist of bold elements on the table above.

A={$\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix}$, $\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 3\end{pmatrix}$, $\begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 2\end{pmatrix}$, $\begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 3\end{pmatrix}$, $\begin{pmatrix}1 & 2 & 3\\ 2 & 2 & 3\end{pmatrix}$, $\begin{pmatrix}1 & 2 & 3\\ 1 & 1 & 1\end{pmatrix}$}

A is an independent set of T_3. The cardinality of A is

|A|=[S(3, 2)-(3-1)](3-1)!+3+1=6. Hence $r_4(T_3) \geq 6$

Remark

Howie and Riberio proved that the large rank of T_n is n^2-(1/2)n!+1. For T_3, r_5 $(T_3)=3^3$-(1/2)3!+1=25. Since $r_1(S) \leq r_2(S) \leq r_3(S) \leq r_4(S) \leq r_5(S)$ and $6 \leq r_4(T_3) \leq 25 = r_5(T_3)$, our lower bound is true.

Thus all five ranks of T_n, $r_1(T_n) = 1 \leq r_2(T_n) = 3 \leq r_3(T_n) = 3 \leq r_4(T_n) = 6 \leq r_5(T_n) = 25$ were calculated.

Corollary

The independent set of $T_n$is not unique. There are $\binom{n}{n-1}$ independent sets. Here $\binom{n}{n-1}$ is the number of distinct $\mathcal{L}$-classes in D_{n-1}.

From the above example, there are $\binom{3}{3-1}$=3 independent sets of T_3. These are below for $L_{1,2}$, $L_{1,3}$ and$L_{2,3}$ respectively.

A_1={$\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix}$, $\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 2\end{pmatrix}$, $\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 1\end{pmatrix}$, $\begin{pmatrix}1 & 2 & 3\\ 1 & 1 & 2\end{pmatrix}$, $\begin{pmatrix}1 & 2 & 3\\ 2 & 2 & 1\end{pmatrix}$, $\begin{pmatrix}1 & 2 & 3\\ 3 & 3 & 3\end{pmatrix}$}

A_2={$\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix}$, $\begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 3\end{pmatrix}$, $\begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 1\end{pmatrix}$, $\begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 3\end{pmatrix}$, $\begin{pmatrix}1 & 2 & 3\\ 1 & 1 & 3\end{pmatrix}$, $\begin{pmatrix}1 & 2 & 3\\ 2 & 2 & 2\end{pmatrix}$}

A_3={$\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix}$, $\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 3\end{pmatrix}$, $\begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 2\end{pmatrix}$, $\begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 3\end{pmatrix}$, $\begin{pmatrix}1 & 2 & 3\\ 2 & 2 & 3\end{pmatrix}$, $\begin{pmatrix}1 & 2 & 3\\ 1 & 1 & 1\end{pmatrix}$}

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