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RESEARCH ARTICLE

KÄHLER MANIFOLDS AS TARGET SPACE OF SUPERGRAVITY THEORIES: CHARACTERISTICS FOR D=4, N=1 THEORY

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ABSTRACT

In this paper the important role of Kähler manifolds in the process of building supergravity theories is considered. After a brief introduction connected to particle physics and field theory, details about almost complex and complex structures, so as about the differential geometry of Kähler manifolds, will be done. Focusing then on D=4, N=1 supergravity, technical tools related to the coupling of pure D=4, N=1 supergravity to scalar and vector multiplets are also given.

1. INTRODUCTION

The geometrical aspects of field theories have always aroused great interest, because of their intrinsic importance and for the positive impact on the studied theories. In particular, the link between σ - models and geometry has been introduced in 1960 by Gell-Mann and Lévy for studying the interaction between pions and nucleons (Gell-Mann*et al.,* 1960). They understood that fields are maps among manifolds, and the action functional is created with the only metric of the target manifold. Around 1970 Zumino clearly understood the importance of Kähler manifolds as requirement of target space for obtaining σ - models with a supersymmetry group (Zumino, 1979). Later it has been found that the extension of supersymmetry to a higher number of generators requires additional geometric structures; σ - models with more complicated geometric structures have been introduced, which determine different gauge choices.The multi-structured manifolds are endowed with one or more tensor fields, linked by appropriate conditions of mutual compatibility. An important example is precisely provided by Kähler manifolds, where the Riemannian and complex structures are coupled in order to obtain a symplectic structure.Considering the pure supergravity, we treat only with the supermultiplet of graviton. The presence of spin zero states involves new structures (σ - models) and new physical consequences (as the super-Higgsphenomenon). Field theories of particles with spin 1/2 and zero, which have some global symmetries under a transformations group G, can be made invariant under local gauge transformations if they are coupled to the Yang-Mills theory of the same group G. It is the case of pure N = 1, D = 4 supergravity, as simple example, with the scalar multiplet $(1/2\,, 0^+, 0^-)$.

If the Lagrangian contains also a non-trivial potential term $W(\varphi)$, as function of scalar fields φ, the Higgs phenomenon is present. This happens if $W(\varphi)$ admits extremes $\varphi = \varphi_0$:

$$\frac{\partial W(\varphi)}{\partial \varphi}_{\varphi=\varphi_0} = 0 \;,(1)$$

which break the symmetry of G to a subgroup $H \subset G$.

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In this case the gauge fields $A^k{}_\mu$, corresponding to the broken generators of *G*, become massive "eating" the freedom degrees of a corresponding number of scalar fields. The spontaneous breaking of gauge symmetries via the Higgs mechanism is essential for applications of the Yang-Mills theory to the description of the interactions among elementary particles.It can thus be suggested the possibility of a spontaneous breaking of local supersymmetry, i.e. a super-Higgs phenomenon. Theories including scalar fields φ and having a scalar potential $W(\varphi)$ can have extremes $\varphi = \varphi_0$ which are not invariant under supersymmetry, and therefore break it spontaneously. In this case, gravitinos corresponding to the generators of supersymmetry become massive "eating" the freedom degrees of spin 1/2 fields, partners of scalar fields φ, which give a non-zero vacuum expectation value; this possibility is the basis of the phenomenological applications of supergravity. For having indeed consistency with phenomenology, supersymmetry must occur in Nature as a spontaneously broken symmetry.

One of the main reasons for consider supergravity in the context of particle physics is its ability to solve, in the spontaneously broken version, the problem of "gauge hierarchy", i.e. to stabilize the relation between the mass scale of the weak interactions and grand unification:

$$\frac{M_W}{M_X} \cong 10^{-12} \text{ .(2)}$$

Differently from the Yang-Mills theory, in supergravity models $W(\varphi)$ is a consequence of supersymmetry, it is not "imposed from outside". Moreover the same scalar field, which breaks supersymmetry, may also break bosonic gauge symmetries. One of the key points is that the scalar fields φ^i, regardless of the belonging multiplet, can be regarded as the coordinates of a convenient differentiable manifold $\mathscr{M}$, endowed with a Riemannian metric $g_{ij}(\varphi)$. The choice of the multiplet, of the number *N* of supersymmetries and of the space-time dimension *D* is reflected in the geometric properties of the manifold$\mathscr{M}$ (Isham, 1999; Drees, 1996).The scalar manifolds $\mathscr{M}$ are normally Kähler manifolds. This fact has been proved in general considering the constraints that supersymmetry imposes on supersymmetricLagrangians involving Wess-Zuminomultiplets: the couplings described by these Lagrangians are compatible with supersymmetry only if the scalar fields $(0^+, 0^-)$ parameterize a complex Riemannian manifold with a Kähler structure. In particular, the complex field:

$$z^i = A^i + i\, B^i \ , \quad (3)$$

is introduced, in such a way to locally underline the complex structure of the manifold. In the following we consider technical peculiarities of the differential geometry of Kähler manifolds, as well as Kähler geometry details for coupling of scalar and vector multiplets to *D* = 4, *N* = 1 pure supergravity(Bilal, 2001; Di Sia, 2014).

2. ALMOST COMPLEX AND COMPLEX STRUCTURES ON A 2n-DIMENSIONAL MANIFOLD

Let $\mathscr{M}$be a 2*n*-dimensional manifold, with *T*($\mathscr{M}$) as tangent space and *T**($\mathscr{M}$) as cotangent space. The vectors of *T*($\mathscr{M}$) are the linear differential operators:

$$\vec{t} = t^\alpha(\varphi)\vec{\partial}_\alpha = t^\alpha(\varphi)\frac{\partial}{\partial \varphi^\alpha} \text{ .(4)}$$

The vectors of *T**($\mathscr{M}$) are the differential 1-forms:

$$\omega = d\varphi^\alpha \ \omega_\alpha(\varphi) \ \text{ .(5)}$$

We can consider linear operators *L* on *T*($\mathscr{M}$):

$$L : T(\mathscr{M}) \to T(\mathscr{M}),(6)$$

such that:

$$\forall \vec{t} \in T(\mathscr{M}): \ \vec{t} L \in T(\mathscr{M});(7)$$

$$\forall \alpha, \beta \in \mathbb{C}, \ \forall \vec{t}_1, \vec{t}_2 \in \ T(\mathscr{M}): \ (\alpha \vec{t}_1 + \beta \vec{t}_2) L = \ \alpha \vec{t}_1 L + \beta \vec{t}_2 L \text{ .(8)}$$

In every local chart, L is represented by a mixed tensor $L_\alpha{}^\beta(\varphi)$:

$$\vec{t}\,L = t^\alpha(\varphi)\,L_\alpha{}^\beta(\varphi)\,\vec{\partial}_\beta \,.(9)$$

The action of L is naturally translated on the cotangent space:

$$L : T^*(\mathscr{M}) \rightarrow T^*(\mathscr{M}),(10)$$

$$L\,\omega = d\varphi^\alpha\, L_\alpha{}^\beta(\varphi)\,\omega_\beta(\varphi)\,.(11)$$

A $2n$-dimensional manifold $\mathscr{M}$ is said “almost complex” if it has an “almost complex structure”. An “almost complex structure” is a linear operator:

$$J : T(\mathscr{M}) \rightarrow T(\mathscr{M}),(12)$$

that satisfies the property:

$$J^2 = -1.(13)$$

In every local chart the operator J is represented by a tensor $J_\alpha{}^\beta(\varphi)$ such that:

$$J_\alpha{}^\beta(\varphi)\,J_\beta{}^\gamma(\varphi) = -\delta_\alpha{}^\gamma\,.(14)$$

Moreover, through a suitable change of basis, in every point $P \in \mathscr{M}$ it is possible to reduce $J_\alpha{}^\beta(\varphi)$ in the form:

$$J_\alpha{}^\beta(\varphi) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.(15)$$

A local frame with J given by (15) is said “well adapted frame”. Naming:

$$\vec{e}_\alpha = \vec{\partial}_\alpha = \frac{\partial}{\partial\varphi^\alpha}\,,(16)$$

the basis vectors of the well adapted frame, one has:

$$\vec{e}_\alpha\,J = -\vec{e}_{\alpha+n}\ ;\ \alpha \le n\,,(17)$$

$$\vec{e}_\alpha\,J = \vec{e}_{\alpha-n}\ ;\ \alpha > n\,.(18)$$

Introducing a Latin index $i = 1,\ldots, n$, we define the complex vectors:

$$\vec{E}_i = \vec{e}_i - i\,\vec{e}_{i+n}\,,(19)$$

$$\vec{E}_{i*} = \vec{e}_i + i\,\vec{e}_{i+n}\,,\ (20)$$

obtaining:

$$\vec{E}_i\,J = i\,\vec{E}_i\,,\ (21)$$

$$\vec{E}_{i*}\,J = -i\,\vec{E}_{i*}\,.(22)$$

The tangent vectors $\vec{E}_i$ are the partial derivatives along the complex coordinates:

$$z^i = \varphi^i + i\,\varphi^{i+n}\,,\ (23)$$

and $\vec{E}_{i*}$ are the partial derivatives along the complex conjugates $\bar{z}^{i*}$:

$$\vec{E}_i = \vec{\partial}_i = \frac{\partial}{\partial z^i}\,,(24)$$

$$\vec{E}_{i*} = \vec{\partial}_{i*} = \frac{\partial}{\partial \bar{z}^{i*}}\,.(25)$$

The existence of the almost complex structure ensures that in every point $P \in \mathscr{M}$ it is possible to replace the $2n$ real coordinates with n complex coordinates, corresponding to a "well adapted frame". Furthermore, each pair of well adapted frame is linked to each other by means of a coordinate transformation, which is a holomorphic function of the complex coordinates (Castellani*et al.,*1991).

3. KÄHLER AND HERMITIAN METRICS

A "metric" g is a scalar-values symmetric bilinear functional on $T(\mathscr{M}) \otimes T(\mathscr{M})$:

$$g\!:\ T(\mathscr{M}) \otimes T(\mathscr{M}) \ \rightarrow\ \mathbb{R};(26)$$

$$g(\vec{u},\vec{w}) = g_{\alpha\beta}\, u^\alpha\, w^\beta\,.\ (27)$$

It is: $g_{\alpha\beta}(x) = g_{\beta\alpha}(x)$; u^α and w^β are the components of the tangent vectors $\vec{u}$ and $\vec{w}$ .Let $\mathscr{M}$ be a $2n$-dimensional manifold with an almost complex structure J. A metric g on $\mathscr{M}$ is said "hermitian with respect to J" if:

$$g(\vec{u}\,J, \vec{w}\,J) = g(\vec{u},\vec{w})\,.(28)$$

An almost complex manifold equipped with a hermitian metric g is said "almost complex Hermitian manifold". $g(\vec{u},\vec{w})$ can be written as follows:

$$g(\vec{u},\vec{w}) \ = \ g_{\alpha\beta}\, u^\alpha\, w^\beta \ = \ g_{ij}\, u^i\, w^j + g_{i*j}\, u^{i*}\, w^j + g_{ij*}\, u^i\, w^{j*} + g_{i*j*}\, u^{i*}\, w^{j*}\,.\ \ (29)$$

The following properties hold:

a) reality of $g(\vec{u},\vec{w})$:

$$g_{ij} = (g_{i*j*})^*\,,(30)$$

$$g_{i*j} = (g_{ij*})^*\,;(31)$$

b) symmetry of $g(\vec{u},\vec{w})$:

$$g_{ij} = g_{ji}\,,(32)$$

$$g_{ij*} = g_{j*i}\,;(33)$$

c) hermiticity of $g(\vec{u},\vec{w})$:

$$g_{ij} = g_{i*j*} = 0\,.\ \ (34)$$

The metric g is represented by a Hermitian matrix g_{ij*}:

$$g_{ij*} = (g_{ji*})^* = g_{j*i} . \quad (35)$$

The two-form:

$$\omega = i\, g_{ij*}\, dz^i \wedge d\bar{z}^{j*} .(36)$$

is said “Kähler form”.

A Hermitian metric on a complex manifold $\mathscr{M}$ is said “Kähler metric” if the associated 2-form is closed:

$$d\omega = 0 \ . \quad (37)$$

A “Kähler manifold” is a Hermitian complex manifold endowed with a Kähler metric.

4. DIFFERENTIAL GEOMETRY OF KÄHLER MANIFOLDS

The general solution of Eq.(37) in every local chart is given by:

$$g_{ij*} = \partial_i\, \partial_{j*}\, G\ , \quad (38)$$

where $G = G^* = G(z, \bar{z})$ is a real function of z^i and $\bar{z}^{i*}$. This function is said “Kähler potential“ and is defined at less of the real part of a holomorphic function $f(z)$.
According that Greeks indices span both i and i*, the Riemannian affine connection associated to Kähler metric $g_{\alpha\beta} = g_{ij*}$ presents the form:

$$\begin{Bmatrix} \alpha \\ \beta\gamma \end{Bmatrix} = \frac{1}{2} g^{\alpha\mu} \left(\partial_\beta\, g_{\gamma\mu} + \partial_\gamma\, g_{\beta\mu} - \partial_\mu\, g_{\beta\gamma} \right), (39)$$

with:

$$g^{\alpha\mu}\, g_{\beta\mu} = \delta^\alpha{}_\beta\ . \quad (40)$$

It holds also:

$$\begin{Bmatrix} i \\ j\,k \end{Bmatrix} = g^{im^*}\, \partial_j\, g_{km^*}\ ;(41)$$

$$\begin{Bmatrix} i^* \\ j^*\,k^* \end{Bmatrix} = \begin{Bmatrix} i \\ j\,k \end{Bmatrix}^* \ ;(42)$$

$$\begin{Bmatrix} i \\ j\,k^* \end{Bmatrix} = \begin{Bmatrix} i \\ j^*\,k^* \end{Bmatrix} = 0\ ;(43)$$

$$\begin{Bmatrix} i^* \\ j^*\,k \end{Bmatrix} = \begin{Bmatrix} i^* \\ j\,k \end{Bmatrix} = 0\ .(44)$$

The covariant differential of an object v^i which transforms as a covariant vector is given by:

$$\nabla v^i = dv^i + \begin{Bmatrix} i \\ j\,k \end{Bmatrix} dz^j\, v^k\ ,(45)$$

and the Riemannian Kähler curvature by:

$$\nabla^2 v^i = R^i{}_j \, v^j \; ;(46)$$

$$R^i{}_j = R_{m^* n}{}^i{}_j \, d\,\bar{z}^{m^*} \wedge d\,z^n \; ; (47)$$

$$R_{m^* n}{}^i{}_j = \partial_{m^*} \begin{Bmatrix} i \\ n\,j \end{Bmatrix} .(48)$$

Using the relation:

$$\begin{Bmatrix} \alpha \\ \beta\,\alpha \end{Bmatrix} = \partial_\beta \left(\ln \sqrt{g}\right),(49)$$

we can write the Ricci tensor as (Castellani*et al.,*1990; Craps*et al.,*1997; Mohaupt, 2000):

$$R_{m^* n} = R_{m^* n}{}^i{}_i = \partial_{m^*} \begin{Bmatrix} i \\ n\,i \end{Bmatrix} = \partial_{m^*} \partial_n \left(\ln \sqrt{g}\right),(50)$$

with:

$$g = \det\left|g_{\alpha\beta}\right| = \left(\det\left|g_{ij^*}\right|\right)^2 .(51)$$

5. ON KÄHLER GEOMETRY FOR COUPLING OF SCALAR MULTIPLETS TO PURE *D*=4, *N*=1 SUPERGRAVITY

In pure N=1 supergravity we work with the vierbein V^a, the "gravitino" ψ and the spin connection ω^{ab}. From a particles point of view, V^a, ψ and ω^{ab} describe N=1 gravitational multiplet(2, 3/2). We now wants to couple this multiplet to nWess-Zuminomultiplets(1/2, 0^+, 0^-), described by the set of 0-forms (Λ^i, A^i, B^i) (i = 1, ….., n), with A^i and B^i a real scalar and a real pseudo-scalar respectively, and Λ^i a Majoranaspinor.
In this regard it is possible to introduce a set of complex fields z^i:

$$z^i = A^i + i\,B^i \quad (\bar{z}^{i^*} \stackrel{def}{=} (z^i)^* = A^i - i\,B^i),(52)$$

considering them as coordinates of a complex n-dimensional manifold $\mathscr{M}$, to which we assign aKähler structure. On $\mathscr{M}$ the Kähler potential:

$$G = G(\,z^i, \bar{z}^{i^*}); \quad G = G^*,(53)$$

is introduced, and also the chiral projections of spinors Λ^i and ψ :

$$\Lambda^i = \chi^i + \chi^{i^*}; \; \chi^i = \frac{1+\gamma_5}{2}\Lambda^i; \; \chi^{i^*} = \frac{1-\gamma_5}{2}\Lambda^i \quad ;(54)$$

$$\gamma_5\,\chi^i = \chi^i; \; \gamma_5\,\chi^{i^*} = -\chi^{i^*}; \; \chi^{i^*} = C\,\gamma_0{}^T\,(\chi^i)^* ;(55)$$

$$\psi = \psi_\bullet + \psi^\bullet; \; \psi_\bullet = \frac{1+\gamma_5}{2}\psi; \; \psi^\bullet = \frac{1-\gamma_5}{2}\psi \quad ; (56)$$

$$\gamma_5\,\psi_\bullet = \psi_\bullet; \; \gamma_5\,\psi^\bullet = -\psi^\bullet; \; \psi^\bullet = C\,\gamma_0{}^T\,(\psi_\bullet)^* ;(57)$$

$$\overline{\psi}^{\bullet} = (\psi_{\bullet})^{+}\,\gamma_0 = \psi^{+}\,(\frac{1+\gamma_5}{2})\,\gamma_0 = \overline{\psi}\,(\frac{1-\gamma_5}{2}) \;;(58)$$

$$\overline{\psi}_{\bullet} = (\psi^{\bullet})^{+}\,\gamma_0 = \psi^{+}\,(\frac{1-\gamma_5}{2})\,\gamma_0 = \overline{\psi}\,(\frac{1+\gamma_5}{2}) \;;(59)$$

$$\overline{\chi}^{i} = (\chi^{i^*})^{+}\,\gamma_0 = \overline{\Lambda}^{i}\,(\frac{1+\gamma_5}{2}) \;; \quad (60)$$

$$\overline{\chi}^{i^*} = (\chi^{i})^{+}\,\gamma_0 = \overline{\Lambda}^{i}\,(\frac{1-\gamma_5}{2}) \;.(61)$$

The coupling of scalar multiplets to pure supergravity corresponds to build a cross-section of the fiber bundle $B(R^{4/4}, \mathscr{M})$, which has the N=1 superspace$R^{4/4}$ as support space and the Kähler manifold $\mathscr{M}$ as fiber. The z coordinate is a superfield $z^i = z^i\,(x,\theta)$, therefore at every point $(x,\theta)\in R^{4/4}$ of the support we associate a point $z^i \in \mathscr{M}$ of the fiber. Expanding dz^i in the (V , ψ) basis, we can write:

$$dz^i = Z^i{}_a\,V^a + \overline{\chi}^i\,\psi_{\bullet} = Z^i{}_a\,V^a + \overline{\psi}_{\bullet}\,\chi^i\,, \quad (62)$$

$$d\overline{z}^{i^*} = \overline{Z}^{i^*}{}_a\,V^a + \overline{\chi}^{i^*}\,\psi^{\bullet} = \overline{Z}^{i^*}{}_a\,V^a + \overline{\psi}^{\bullet}\,\chi^{i^*}\,, \quad (63)$$

having considered the rheonomic condition by writing that the “out” component of dz is χ , i.e. the spin 1/2 field; therefore $Z^i{}_a$ is a vector field and $\overline{\chi}^i$ is a left-handed spinor field. The action of the Kähler transformation on fermionic fields can be considered as a chiral rotation. The Kähler connection is defined as:

$$Q = \frac{1}{2i}(\partial_i\,G\,dz^i - \partial_{i^*}\,G\,d\overline{z}^{i^*})\,. \quad (64)$$

The curvature of the Kähler connection is the 2-form K defined as:

$$K = dQ = i\,g_{ij^*}\,dz^i \wedge d\overline{z}^{j^*}\,. \quad (65)$$

Using Eqs (62, 63) and defining the quantities:

$$K_{ab} = i\,g_{ij^*}\,Z^i{}_{[a}\,\overline{Z}^{j^*}{}_{b]}, \quad (66)$$

$$T_a = \overline{\chi}^i\,\gamma_a\,\chi^{j^*}\,g_{ij^*}\,, \quad (67)$$

$$\Sigma^a{}_{\bullet} = i\,g_{ij^*}\,\chi^i\,\overline{Z}^{j^* a}\,, \quad (68)$$

$$\Sigma^{\bullet}{}_a = (\Sigma_{\bullet a})^C = -i\,g_{i^* j}\,\chi^{i^*}\,Z^j{}_a\,, \quad (69)$$

it is possible to write:

$$K = K_{ab} \wedge V^a \wedge V^b + \frac{i}{2}T_a\,\overline{\psi}_{\bullet} \wedge \gamma^a\,\psi^{\bullet} + \overline{\psi}_{\bullet}\,\Sigma^a{}_{\bullet} \wedge V_a + \overline{\psi}^{\bullet}\,\Sigma^{\bullet}{}_a \wedge V^a \;.(70)$$

The exterior derivatives of the matter fields z^i and χ^i are the analogue of the curvatures R^{ab}, T^a e ρ of the supergravity fields. More precisely, it is useful to define as “curvature” of χ the covariant derivative $\nabla\chi^i$, which is covariant with respect to

Lorentz, Kähler transformations, and to coordinate transformations on Kähler manifold. In general, all covariant derivatives of fermions contain the Kähler connection, for being covariant under Kähler transformations.
Therefore the set of curvatures of supergravity and of Wess-Zuminomultiplets are given by:

$$R^a = \mathscr{D}V^a - i\overline{\psi}^\bullet \wedge \gamma^a \psi_\bullet ; \quad (\mathscr{D}V^a \equiv dV^a - \omega^{ab} \wedge V_b);(71)$$

$$R^{ab} = d\omega^{ab} - \omega^a{}_c \wedge \omega^{cb} \equiv \mathscr{R}^{ab};(72)$$

$$\rho_\bullet = \nabla \psi_\bullet ; \quad (73)$$

$$R(z)^i \overset{def}{=} dz^i = dz^i ; \quad (74)$$

$$R(\chi)^i \overset{def}{=} \nabla \chi^i = \nabla \chi^i ,(75)$$

where:

$$\nabla \psi_\bullet = d\psi_\bullet - \frac{1}{4}\omega^{ab} \wedge \gamma_{ab} \psi_\bullet + \frac{i}{2} Q \wedge \psi_\bullet ; (76)$$

$$\nabla \chi^i = d\chi^i - \frac{1}{4}\omega^{ab} \wedge \gamma_{ab} \chi^i + \left\{ \begin{matrix} i \\ j\,k \end{matrix} \right\} dz^j \chi^k + \frac{i}{2} Q \chi^i .(77)$$

The Bianchi identities of supergravity added to Wess-Zumino are given by (Di Sia, 2014; Castellani*et al.,*1991; Freedman and Van Proeyen, 2012; Nastase, 2012):

$$\mathscr{D}R^a + R^{ab} \wedge V_b + i\overline{\rho}^\bullet \wedge \gamma^a \psi_\bullet - i\overline{\psi}^\bullet \wedge \gamma^a \rho_\bullet = 0 ;(78)$$

$$\mathscr{D}R^{ab} = 0 ;(79)$$

$$\nabla \rho_\bullet + \frac{1}{4} R^{ab} \wedge \gamma_{ab} \psi_\bullet - \frac{i}{2} K \wedge \psi_\bullet = 0 ; \quad (80)$$

$$d\,d\,z^i = 0 ; \quad (81)$$

$$\nabla\nabla \chi^i + \frac{1}{4} R^{ab} \wedge \gamma_{ab} \chi^i + \frac{i}{2} K \chi^i - R_{m^* j}{}^i{}_k \, d\bar{z}^{m^*} \wedge dz^j \chi^k = 0 . \quad (82)$$

6. ON KÄHLER GEOMETRY FOR COUPLING OF VECTOR MULTIPLET TO PURE *D*=4, *N*=1 SUPERGRAVITY COUPLED WITH SCALAR MULTIPLETS

The scalar multiplets contain quarks, leptons and Higgs particles together with their superpartners. The gauge bosons, conversely, belong to vector multiplets (1, 1/2). Similarly to the ordinary Yang-Mills theory, the role of vector multiplets is to "make local" some groups of global symmetries of the matter Lagrangian. The global bosonic symmetries are in bijective correspondence with the isometries of the Kähler metric $g_{ij^*}(z,\bar{z})$, which satisfy the additional requirement of maintaining invariant the Kähler potential $G(z,\bar{z})$.
If indeed $K^i{}_{(\alpha)}(z)$ is a basis of holomorphic Killing vectors for the metric $g_{ij^*}(z,\bar{z})$, the holomorphicity means:

$$\partial_{j^*} K^i{}_{(\alpha)}(z) = 0 \;\Rightarrow\; \partial_j K^{i^*}{}_{(\alpha)}(\bar{z}) = 0 ; \quad (83)$$

$$K^{i^*}{}_{(\alpha)} = (K^i{}_{(\alpha)})^* . \quad (84)$$

The $K^i{}_{(\alpha)}(z)$ vectors are the generators of transformations of infinitesimal holomorphic coordinates:

$$\delta z^i = \varepsilon^\alpha K^i{}_{(\alpha)}(z) \text{ ,(85)}$$

which maintain invariant the metric $g_{ij^*}(z,\bar{z})$.
The vector fields:

$$\vec{K}_{(\alpha)} = K^i{}_{(\alpha)} \vec{\partial}_i , \quad (86)$$

associated to such Killing vectors close a Lie algebra:

$$\left[\vec{K}_{(\alpha)} , \vec{K}_{(\beta)}\right] = h_{\alpha\beta}{}^\gamma \vec{K}_{(\gamma)} \text{ ,(87)}$$

and vectors may be normalized in such a way that the structure constants are fully antisymmetric:

$$h_{\alpha\beta}{}^\gamma = h_{\alpha\beta\gamma} = h_{[\alpha\beta\gamma]} \text{ .(88)}$$

As the metric $g_{ij^*}(z,\bar{z})$ is the derivative of other fundamental objects, so the Killing vectors in a Kähler manifold are the derivatives of a convenient prepotential:

$$\vec{K}^i{}_{(\alpha)} = i\, g^{ij^*} \partial_{j^*} \mathscr{P}_{(\alpha)} ; \quad \mathscr{P}^*{}_{(\alpha)} = \mathscr{P}_{(\alpha)} . \quad (89)$$

We can therefore define a Killing vector finding a real function $\mathscr{P}_{(\alpha)}$ such that $i\, g^{ij^*} \partial_{j^*} \mathscr{P}_{(\alpha)}$ is holomorphic.
The infinitesimal transformation of holomorphic coordinates extended to fermions is an invariance of the part of Langrangian, which survive to the limit "$e \to 0$", because under an isometry the Kähler potential is not invariant, but changes for a Kähler transformation

$$G(z+\delta z, \bar{z}+\delta \bar{z}) = G(z,\bar{z}) + \varepsilon^\alpha \operatorname{Re} f_\alpha(z) , \quad (90)$$

which can be compensated in this part of the Lagrangian by another Kähler transformation. The form of the isometric transformation on fermions is therefore:

$$\delta\chi^i = \varepsilon^\alpha \partial_j K^i{}_{(\alpha)}(z) \chi^j - \frac{i}{2} \varepsilon^\alpha f_\alpha(z) \chi^i ; (91)$$

$$\delta\psi_\bullet = \frac{i}{2} \varepsilon^\alpha \operatorname{Im} f_\alpha(z) \psi_\bullet . \quad (92)$$

Even the part of the Lagrangian proportional to "e" (which contains the mass term of gravitino, the "non-diagonal" mass term, the mass term of spin 1/2 and the potential term of the scalar field) is not invariant under the isometry $K^i{}_{(\alpha)}(z)$, unless the compensating Kähler transformation is zero:

$$f_\alpha(z) = 0 \text{ .(93)}$$

We therefore consider holomorphic vectors $K^i{}_{(\alpha)}(z)$ that satisfy the most restrictive condition of maintaining invariant the Kähler potential:

$$\partial_i G\, K^i{}_{(\alpha)} + \partial_{i^*} G\, K^{i^*}{}_{(\alpha)} = 0 . \quad (94)$$

In particle physics applications it is possible to have the situation in which the Killing vector is a linear function in z:

$$K^i{}_{(\alpha)} = (T_\alpha)^i{}_j\, z^j \;\Rightarrow\; \delta z^i = \varepsilon^\alpha\, (T_\alpha)^i{}_j\, z^j \;. \quad (95)$$

In this case, Eq. (91) becomes:

$$\delta\chi^i = \varepsilon^\alpha\, (T_\alpha)^i{}_j\, \chi^j \,. \,(96)$$

In the case of linear isometries, the prepotential of Killing vectors is expressed in terms of the first derivative of the Kähler potential (Di Sia, 2014; Castellani*et al.,*1991; Castellani *et al.,*1990; Freedman and Van Proeyen, 2012; Nastase, 2012):

$$\mathscr{P}_{(\alpha)} = -i\,\partial_i G\;\; (T_\alpha)^i{}_j\, \chi^j \,.(97)$$

7. CONCLUSION

In this paper it has been underlined the positive contribution of Kähler manifolds in the process of building supergravity theories. Supersymmetry imposes particular constraints on the geometry of the scalar manifold $\mathscr{M}$, on which the scalar fields are considered as coordinates. In *D*=4, *N*=1 theories the scalar manifold for chiral multiplets$\mathscr{M}$is restricted to be a Kähler manifold. The choice of the multiplet, the number *N* of supersymmetries and the space-time dimension *D* is reflected in geometric properties of the manifold$\mathscr{M}$. Multi-structured manifolds are endowed with one or more tensor fields, linked by appropriate conditions of mutual compatibility. Kähler manifolds constitute a significant example, with Riemannian and complex structures coupled in order to obtain a symplectic structure.
From the work of Gell-Mann, Lévy and Zumino it has been clearly understood the importance of Kähler manifolds as target space requirement for having models with supersymmetry group. Later it has been found that the extension of supersymmetry to higher numbers of generators requires additional geometric structures.The general geometric form of supergravity coupled to both scalar and vector multiplets before the supersymmetry breaking, i.e. the theory of heterotic string, of which the *N*=1 supergravity is the effective theory in 4 dimensions, is in agreement with the results obtained by means of the superconformal tensor calculus within the components approach (Di Sia, 2014; Castellani*et al.,*1991; Castellani *et al.,*1990; Di Sia, 2013; Di Sia, 2015).

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