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FIVE SIMULTANEOUS FOURIER SERIES EQUATIONS

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ABSTRACT

There has been a lot of work on dual, triple and quadruple series equations involving different polynomials. Due to the importance of these series equations in finding the solutions of various mixed boundary value problems of elasticity, electrostatics and other fields of mathematical physics, a number of researchers took interest in finding the series solution as well as developing and investigating new classes of series equations. There was almost no research work on five series equations until Dwivedi and Pandey taken it into consideration. They solved certain five series equations involving generalized Bateman K-functions, series of Jacobi and Laguerre and the product of 'r' generalized Bateman K-function. In the subsequent years Dwivedi and singh [5, 6], Dwivedi and Chandel [1], obtained the solution of five series equations involving generalized Bateman K-function and Jacobi polynomials respectively. In the present paper, we have considered five series equations involving series of Jacobi polynomials, which are extensions of quadruple series and untouched till date.

INTRODUCTION

There has been a lot of work on dual, triple and quadruple series equations involving different polynomials. Due to the importance of these series in finding the solutions of various mixed boundary value problems of elasticity, electrostatics and other fields of mathematical physics, a number of researchers took interest in finding the series solution as well as developing and investigating new classes of series equations. There was almost no research work on five series equations until Dwivedi and Pandey [2, 3, 4] taken it into consideration. They solved certain five series equations involving generalized Bateman K-functions, series of Jacobi and Leguerre and the product of 'r' generalized Bateman K-function. In the subsequent years Dwivedi and singh [5, 6], Dwivedi and Chandel [1], obtain the solution of five series equations involving generalized Bateman K-function and Jacobi polynomials respectively. In the present paper, we have considered five series equations involving series of Jacobi polynomials which are extensions of quadruple series and untouched till date.

FIVE SIMULTANEOUS FOURIER SERIES EQUATIONS INVOLVING JACOBI POLYNOMIALS

The solution of five series equations involving series of Jacobi polynomials is obtained by reducing them to Fredholm integral equations of the second kind in one independent variable. Dual, triple, quadruple and five series equations involving series of Jacobi polynomials can be change into the series involving ultra-spherical polynomials or Fourier cosine series by small amendment to the original one. These latter series equations play an important role in solving the mixed boundary value problems, when we consider the distribution of stresses in the interior of an infinitely long strip containing three Griffith cracks situated on a line perpendicular to the boundary lines of the strip. Here we are concerned only with five series equations involving series of Jacobi polynomials which are extensions of quadruple series considered by Dwivedi and Singh.

THE EQUATIONS

We shall solve the following set of five series equations

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$$\sum_{n=0}^{\infty}\frac{A_n}{\Gamma(n+\alpha+1)\Gamma\left(n+\beta+\frac{3}{2}\right)}P_n^{(\alpha,\,\beta)}(\mathrm{Cos}\theta)=\begin{cases}f_1(\theta), & 0\le\theta<a \quad (1.1)\\ f_3(\theta), & b<\theta<c \quad (1.2)\\ f_5(\theta), & d<\theta<\pi \quad (1.3)\end{cases}$$

$$\sum_{n=0}^{\infty}\frac{A_n}{\Gamma(n+\beta+1)\Gamma\left(n+\alpha+\frac{1}{2}\right)}P_n^{(\alpha,\,\beta)}(\mathrm{Cos}\theta)=\begin{cases}f_2(\theta), & a<\theta<b \quad (1.4)\\ f_4(\theta), & c<\theta<d \quad (1.5)\end{cases}$$

Where $\alpha,\ \beta>-\frac{1}{2}$, and $f_i(\theta),\ (i=1,\ 2,\ 3,\ 4,\ 5)$ are prescribed functions and equations (1.1) to (1.5) are to be solved for unknown coefficients A_n. It is assumed that series (1.1) to (1.5) are uniformly convergent and $f_i(\theta)$ and their derivatives are continuous.

PRELIMINARY RESULTS

In the course of analysis, we require following results:

The Orthogonality Relation for Jacobi Polynomials

$$\int_0^x\left(\mathrm{Sin}\frac{\theta}{2}\right)^{2\alpha}\left(\mathrm{Cos}\frac{\theta}{2}\right)^{2\beta}P_n^{(\alpha,\,\beta)}(\mathrm{Cos}\theta)P_m^{(\alpha,\,\beta)}(\mathrm{Cos}\theta)\mathrm{Sin}\theta d\theta$$

$$=\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{q_n^{(\alpha,\,\beta)}}\delta_{mn} \qquad (2.1)$$

Is valid for $\alpha>-1,\ \beta>-1$

Where δ_{mn} is Kronecker delta and

$$q_n^{(\alpha+\beta)}=\frac{1}{2}n!\{(\alpha+\beta+2n+1)\}\Gamma(\alpha+\beta+n+1)$$

The Series

$$S(u,\ \theta)=\sum_{n=0}^{\infty}=\frac{q_n^{(\alpha,\,\beta)}=\left(n+\alpha+\frac{1}{2}\right)}{\{\Gamma(n+\alpha+1)\}^2\,\Gamma\left(n+\beta+\frac{3}{2}\right)}$$

$$\left(\mathrm{Sin}\frac{u}{2}\right)^{2\alpha}P_n^{(\alpha,\,\beta)}(\mathrm{Cosu})\times P_n^{(\alpha,\,\beta)}(\mathrm{Cos}\theta) \qquad (2.2)$$

$$= \frac{\left(\operatorname{Sin}\frac{\theta}{2}\right)^{-2\alpha}}{\pi} \int_0^{\min(u,\,\theta)} \frac{E(y)dy}{(\operatorname{Cos}y - \operatorname{Cos}u)^{1/2} (\operatorname{Cos}y - \operatorname{Cos}\theta)^{1/2}} \tag{2.3}$$

where

$$E(t) = \left(\operatorname{Sin}\frac{t}{2}\right)^{2\alpha} \left(\operatorname{Cos}\frac{t}{2}\right)^{-2\beta}, \qquad t = (u,\ \theta).$$

We shall use the following two forms of Schlomilch's Integral Equations:

If $f(\theta)$ and $f'(\theta)$ are continuous in a $a \le \theta \le b,$ then the solutions of the integral equations:

$$f(\theta) = \int_a^{\theta} \frac{g(u)du}{(\operatorname{Cos}u - \operatorname{Cos}\theta)^{1/2}} \tag{2.4}$$

and $$f'(\theta) = \int_{\theta}^{b} \frac{g'(u)du}{(\operatorname{Cos}\theta - \operatorname{Cos}u)^{1/2}} \tag{2.5}$$

are $$g(u) = \frac{1}{\pi}\frac{d}{du}\int_a^{u} \frac{f(\theta)\operatorname{Sin}\theta d\theta}{(\operatorname{Cos}\theta - \operatorname{Cos}u)^{1/2}} \tag{2.6}$$

and $$g'(u) = \frac{1}{\pi}\frac{d}{du}\int_u^{b} \frac{f'(\theta)\operatorname{Sin}\theta d\theta}{(\operatorname{Cos}u - \operatorname{Cos}\theta)^{1/2}} \tag{2.7}$$

respectively.

THE SOLUTION

Let us suppose

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(n+\beta+1)\Gamma\left(n+\alpha+\frac{1}{2}\right)} P_n^{(\alpha,\,\beta)}(\operatorname{Cos}\theta) = \begin{cases} g(\theta), & 0 \le \theta < a \quad (3.1) \\ h(\theta), & b < \theta < c \quad (3.2) \\ k(\theta), & d < \theta < \pi \quad (3.3) \end{cases}$$

Using orthogonality relation (2.1) in equations (1.4), (1.5) and (3.1) to (3.3) we get

$$A_n = \frac{\Gamma\left(n+\alpha+\frac{1}{2}\right)}{\Gamma(n+\alpha+1)} q_n^{(\alpha,\,\beta)} \left[\int_0^a g'(u) + \int_a^b f'_2(u) + \int_b^c h'(u) + \int_c^d f'_4(u)\right.$$

$$\left. + \int_d^{\pi} k'(\theta)\right] \left(\operatorname{Sin}\frac{u}{2}\right)^{2\alpha} P_n^{(\alpha,\,\beta)}(\operatorname{Cos}\theta) \times P_n^{(\alpha,\,\beta)}(\operatorname{Cos}u)du \tag{3.4}$$

where

$$g'(u)=\left(\operatorname{Cos}\frac{u}{2}\right)^{2\beta}\operatorname{Sin}u\ g(u),$$

$$f_2'(u)=\left(\operatorname{Cos}\frac{u}{2}\right)^{2\beta}\operatorname{Sin}u\ f_2(u),\ \text{etc.}$$

Substituting the expression for A_n from (3.4) in equations (1.1) to (1.3), we obtain

$$\sum_{n=0}^{\infty}=\frac{q_n^{(\alpha,\beta)}\left(n+\alpha+\frac{1}{2}\right)}{\{\Gamma(n+\alpha+1)\}^2\Gamma\left(n+\beta+\frac{3}{2}\right)}\left[\int_0^a g'(u)+\int_a^b f'_2(u)+\int_b^c h'(u)+\int_c^d f'_4(u)\right.$$

$$\left.+\int_d^{\pi}k'(u)\right]\left(\operatorname{Sin}\frac{u}{2}\right)^{2\alpha}P_n^{(\alpha,\beta)}(\operatorname{Cos}\theta)P_n^{(\alpha,\beta)}(\operatorname{Cos}u)du$$

$$=\begin{cases} f_1(\theta), & 0\le\theta<a \quad (3.5)\\ f_3(\theta), & b<\theta<c \quad (3.6)\\ f_5(\theta), & d<\theta<\pi \quad (3.7)\end{cases}$$

Applying the summation result (2.2) and interchanging the order of integration and summation, we get

$$\left[\int_0^a g'(u)+\int_b^c h'(u)+\int_d^{\pi}k'(u)\right]S(u,\theta)du \quad \begin{cases} P(\theta), & 0\le\theta<a \quad (5.8)\\ Q(\theta), & b<\theta<c \quad (5.9)\\ R(\theta), & d<\theta<\pi \quad (5.10)\end{cases}$$

where

$$P(\theta)=f_1(\theta)-\left[\int_a^b f'_2(u)+\int_c^d f'_4(u)\right]S(u,\theta)du$$

$$Q(\theta)=f_3(\theta)-\left[\int_a^b f'_2(u)+\int_c^d f'_4(u)\right]S(u,\theta)du$$

and $$R(\theta)=f_5(\theta)-\left[\int_a^b f'_2(u)+\int_c^d f'_4(u)\right]S(u,\theta)du$$

Now using the summation result (2.3) in terms of integral in equation (3.8) we obtain

$$\int_0^a g'(u)du\int_0^{\min(u,\,\theta)}\frac{E(y)dy}{(Cosy-Cosu)^{1/2}(Cosy-Cos\theta)^{1/2}}=\pi\left(Sin\frac{\theta}{2}\right)^{2\alpha}P(\theta)$$

$$-\left[\int_b^c h'(u)du\int_0^{\min(u,\,\theta)}\frac{E(y)dy}{(Cosy-Cosu)^{1/2}(Cosy-Cosu)^{1/2}}\right.$$

$$\left.+\int_d^\pi k'(u)du\int_0^{\min(u,\,\theta)}\frac{E(y)dy}{(Cosy-Cosu)^{1/2}(Cosy-Cosu)^{1/2}}\right]$$

$$0\le\theta<a \tag{3.11}$$

Or $$\left.\int_0^\theta g'udu\int_0^u\frac{E(y)dy}{(Cosy-Cosu)^{1/2}(Cosy-Cosu\theta)^{1/2}}\right]$$

$$+\int_0^a g'(u)du\int_0^\theta\frac{E(y)dy}{(Cosy-Cosu)^{1/2}(Cosy-Cos\theta)^{1/2}}=\pi\left(Sin\frac{\theta}{2}\right)^{2\alpha}P(\theta)$$

$$-\left[\int_b^c h'(u)du\int_0^\theta\frac{E(y)dy}{(Cosy-Cosu)^{1/2}(Cosy-Cosu)^{1/2}}\right.$$

$$\left.+\int_d^\pi k'(u)du\int_0^\theta\frac{E(y)dy}{(Cosy-Cosu)^{1/2}(Cosy-Cosu)^{1/2}}\right]$$

$$0\le\theta<a \tag{3.12}$$

Changing the order of integration in the above equation, we get

$$\int_0^\theta\frac{E(y)dy}{(Cosy-Cos\theta)^{1/2}}\int_y^\theta\frac{g'(u)du}{(Cosy-Cosu)^{1/2}}+\int_0^\theta\frac{E(y)dy}{(Cosy-Cos\theta)^{1/2}}$$

$$\times\int_\theta^a\frac{g'(u)du}{(Cosy-Cosu)^{1/2}}=\pi\left(Sin\frac{\theta}{2}\right)^{2\alpha}P(\theta)$$

$$-\left[\int_0^\theta\frac{E(y)dy}{(Cosy-Cos\theta)^{1/2}}\int_b^c\frac{h'(u)du}{(Cosy-Cosu)^{1/2}}\right.$$

$$+\int_0^{\theta}\frac{E(y)dy}{(Cosy-Cos\theta)^{1/2}}\int_d^{\pi}\frac{k'(u)du}{(Cosy-Cosu)^{1/2}}\Bigg]\quad 0\le\theta<a \tag{3.13}$$

Or $$\int_0^{\theta}\frac{E(y)dy}{(Cosy-Cos\theta)^{1/2}}\int_y^{a}\frac{g'(u)du}{(Cosy-Cosu)^{1/2}}=\pi\left(Sin\frac{\theta}{2}\right)^{2\alpha}P(\theta)$$

$$-\Bigg[\int_0^{\theta}\frac{E(y)dy}{(Cosy-Cos\theta)^{1/2}}\int_b^{c}\frac{h'(u)du}{(Cosy-Cosu)^{1/2}}$$

$$+\int_0^{\theta}\frac{E(y)dy}{(Cosy-Cos\theta)^{1/2}}\int_d^{\pi}\frac{k'(u)du}{(Cosy-Cosu)^{1/2}}\Bigg]\quad 0\le\theta<a \tag{3.14}$$

Using results (2.4) and (2.6) in equation (3.14), we obtain

$$E(y)\int_y^{a}\frac{g'(u)du}{(Cosy-Cosu)^{1/2}}=\pi\frac{1}{\pi}\frac{d}{dy}\int_0^{y}\frac{\left(Sin\frac{\theta}{2}\right)^{2\alpha}P(\theta)Sin\theta d\theta}{(Cos\theta-Cosy)^{1/2}}$$

$$-\frac{1}{\pi}\frac{d}{dy}\int_0^{y}\frac{Sin\theta d\theta}{(Cos\theta-Cosy)^{1/2}}\Bigg\{\int_0^{\theta}\frac{E(y)dy}{(Cosy-Cos\theta)^{1/2}}\int_b^{c}\frac{h'(u)du}{(Cosy-Cosu)^{1/2}}$$

$$+\int_0^{\theta}\frac{E(y)dy}{(Cosy-Cos\theta)^{1/2}}\int_d^{\pi}\frac{k'(u)du}{(Cosy-Cosu)^{1/2}}\Bigg\}\quad 0\le\theta<a \tag{5.15}$$

Changing the order of integration of above equation, we get

$$E(y)\int_y^{a}\frac{g'(u)du}{(Cosy-Cosu)^{1/2}}=\frac{d}{dy}\int_0^{y}\frac{\left(Sin\frac{\theta}{2}\right)^{2\alpha}P(\theta)Sin\theta d\theta}{(Cos\theta-Cosy)^{1/2}}$$

$$-\frac{1}{\pi}\Bigg\{\int_b^{c}\frac{h'(u)du}{(Cosy-Cosu)^{1/2}}\frac{d}{dy}\int_0^{y}E(y)dy\int_t^{y}\frac{Sin\theta d\theta}{(Cos\theta-Cosy)^{1/2}(Cost-Cos\theta)^{1/2}}$$

$$+\int_d^{\pi}\frac{k'(u)du}{(Cosy-Cosu)^{1/2}}\frac{d}{dy}\int_0^y E(y)dy\int_t^y\frac{Sin\theta d\theta}{(Cos\theta-Cosy)^{1/2}(Cost-Cos\theta)^{1/2}}\Bigg\}$$

$$0\le\theta<a \tag{3.16}$$

Using the result

$$\int_t^y\frac{Sin\theta d\theta}{(Cos\theta-Cosy)^{1/2}(Cost-Cos\theta)^{1/2}}=\pi \tag{3.17}$$

in equation (3.16) , we get

$$E(y)\int_y^{\alpha}\frac{g'(u)du}{(Cosy-Cosu)^{1/2}}=P_1(y)-\int_b^c\frac{E(y)h'(u)du}{(Cosy-Cosu)^{1/2}}$$

$$-\int_d^{\pi}\frac{E(y)k'(u)du}{(Cosy-Cosu)^{1/2}}\ 0\le\theta<a \tag{3.18}$$

where

$$P_1(y)=\frac{d}{dy}\int_0^y\frac{\left(Sin\frac{\theta}{2}\right)^{2\alpha}P(\theta)Sin\theta d\theta}{(Cos\theta-Cosy)^{1/2}} \tag{3.19}$$

Again using the results (2.5) and (2.7) in equation (3.10), we get

$$E(y)g'(u)=-\frac{1}{\pi}\frac{d}{du}\int_u^a\frac{SinP_1(y)dy}{(Cosu-Cosy)^{1/2}}+\frac{1}{\pi}\frac{d}{du}\int_u^a\frac{E(y)Sin(y)dy}{(Cosu-Cosy)^{1/2}}$$

$$\times\int_b^c\frac{h'(\nu)d\nu}{(Cosy-Cos\nu)^{1/2}}+\frac{1}{\pi}\frac{d}{du}\int_u^a\frac{E(y)Sin(y)dy}{(Cosu-Cosy)^{1/2}}$$

$$\times\int_d^{\pi}\frac{k'(\nu)d\nu}{(Cosy-Cos\nu)^{1/2}}\ 0\le u<a \tag{3.20}$$

Now changing the order of integration and using the result

$$\frac{d}{du}\int_a^u\frac{Sinydy}{(Cosu-Cosy)^{1/2}(Cosy-Cos\nu)^{1/2}}=\frac{Sinu(Cos\nu-Cosa)^{1/2}}{(Cosa-Cosu)^{1/2}(Cos\nu-Cosu)^{1/2}} \tag{3.21}$$

in equation (3.20), becomes

$$E(y)g'(u) = P_2(u) + \frac{1}{\pi}\left[\int_b^c h'(\nu)A(\nu,\ y)d\nu + \int_d^\pi k'(\nu)A(\nu,\ y)d\nu\right]$$

$$0 < u < a \qquad (3.22)$$

where

$$P_2(u) = -\frac{1}{\pi}\frac{d}{du}\int_u^a \frac{\mathrm{Sin}P_1(y)dy}{(\mathrm{Cos}u - \mathrm{Cos}y)^{1/2}} \qquad (3.23)$$

$$A(\nu,\ y) = \frac{E(y)\mathrm{Sin}u(\mathrm{Cos}\nu - \mathrm{Cos}a)^{1/2}}{(\mathrm{Cos}a - \mathrm{Cos}u)^{1/2}(\mathrm{Cos}\nu - \mathrm{Cos}u)} \qquad (3.24)$$

Again using summation result (2.3), in terms of integral in equation (3.9), we obtain

$$\left[\int_b^\theta h'(u) + \int_\theta^c h'(u)\right]\int_0^{\min(u,\ \theta)} \frac{E(y)dudy}{(\mathrm{Cos}y - \mathrm{Cos}u)(\mathrm{Cos}y - \mathrm{Cos}\theta)^{1/2}}$$

$$= \pi\left(\mathrm{Sin}\frac{\theta}{2}\right)^{2\alpha} Q(\theta) - \left[\int_b^a g'(u)du + \int_d^\pi k'(u)du\right]$$

$$\times\int_0^{\min(u,\ \theta)} \frac{E(y)dy}{(\mathrm{Cos}y - \mathrm{Cos}u)(\mathrm{Cos}y - \mathrm{Cos}\theta)^{1/2}}\ b < \theta < c \qquad (3.25)$$

or

$$\int_b^\theta h'(u)du\int_0^u \frac{E(y)dy}{(\mathrm{Cos}y - \mathrm{Cos}u)^{1/2}(\mathrm{Cos}y - \mathrm{Cos}\theta)^{1/2}}$$

$$+\int_\theta^c h'(u)du\int_0^\theta \frac{E(y)dy}{(\mathrm{Cos}y - \mathrm{Cos}u)^{1/2}(\mathrm{Cos}y - \mathrm{Cos}\theta)^{1/2}}$$

$$= \pi\left(\mathrm{Sin}\frac{\theta}{2}\right)^{2\alpha} Q(\theta) - \left[\int_0^a g'(u)du\int_0^u \frac{E(y)dy}{(\mathrm{Cos}y - \mathrm{Cos}u)^{1/2}(\mathrm{Cos}y - \mathrm{Cos}\theta)^{1/2}}\right.$$

$$\left.+\int_d^\pi k'(u)du\int_0^\theta \frac{E(y)dy}{(\mathrm{Cos}y - \mathrm{Cos}u)^{1/2}(\mathrm{Cos}y - \mathrm{Cos}\theta)^{1/2}}\right] b < \theta < c \qquad (3.26)$$

Changing the order of above equation, we get

$$\int_b^\theta \frac{E(y)dy}{(\text{Cos}y-\text{Cos}\theta)^{1/2}}\int_y^c \frac{h'(u)du}{(\text{Cos}y-\text{Cos}u)^{1/2}} = \pi\left(\text{Sin}\frac{\theta}{2}\right)^{2\alpha} Q(\theta)$$

$$+\left[\int_0^b \frac{E(y)dy}{(\text{Cos}y-\text{Cos}\theta)^{1/2}}\int_b^c \frac{h'(u)du}{(\text{Cos}y-\text{Cos}u)^{1/2}}\right.$$

$$+\int_0^a \frac{E(y)dy}{(\text{Cos}y-\text{Cos}\theta)^{1/2}}\int_y^a \frac{g'(u)du}{(\text{Cos}y-\text{Cos}u)^{1/2}}$$

$$\left.+\int_0^\theta \frac{E(y)dy}{(\text{Cos}y-\text{Cos}\theta)^{1/2}}\int_d^\pi \frac{k'(u)du}{(\text{Cos}y-\text{Cos}u)^{1/2}}\right] b<\theta<c \qquad (3.27)$$

Using the results (2.4) and (2.6) in equation (3.27) we get

$$E(y)\int_y^c \frac{h'(u)du}{(\text{Cos}y-\text{Cos}u)^{1/2}} = \frac{d}{dy}\int_b^y \frac{\left(\text{Sin}\frac{\theta}{2}\right)^{2\alpha} Q(\theta)\text{Sin}\theta d\theta}{(\text{Cos}\theta-\text{Cos}y)^{1/2}}$$

$$-\frac{1}{\pi}\frac{d}{dy}\left\{\int_b^y \frac{\text{Sin}\theta d\theta}{(\text{Cos}\theta-\text{Cos}y)^{1/2}}\left[\int_0^b \frac{E(y)dy}{(\text{Cos}y-\text{Cos}\theta)^{1/2}}\int_b^c \frac{h'(u)du}{(\text{Cos}y-\text{Cos}u)^{1/2}}\right.\right.$$

$$+\int_0^a \frac{E(y)dy}{(\text{Cos}y-\text{Cos}\theta)^{1/2}}\frac{g'(u)du}{(\text{Cos}y-\text{Cos}u)^{1/2}}+\int_0^\theta \frac{E(y)dy}{(\text{Cos}y-\text{Cos}\theta)^{1/2}}$$

$$\left.\times\int_d^\pi \frac{k'(u)du}{(\text{Cos}y-\text{Cos}u)^{1/2}}\right] b<\theta<c \qquad (3.28)$$

Changing the order of integration in above equation, we get

$$E(y)\int_y^c \frac{h'(u)du}{(\text{Cos}y-\text{Cos}u)^{1/2}} = Q_1(y)-\frac{1}{\pi}\left[\int_b^c \frac{h'(u)du}{(\text{Cos}y-\text{Cos}u)^{1/2}}\int_0^b E(y)dy\right.$$

$$+\int_t^a \frac{g'(u)du}{(Cost-Cosu)^{1/2}}\int_0^a E(y)dy+\int_d^\pi \frac{k'(u)du}{(Cost-Cosu)^{1/2}}\int_0^\theta E(y)dy\Bigg]$$

$$\times\frac{d}{dy}\int_b^y \frac{Sin\theta d\theta}{\left[(Cos\theta-Cosy)(Cost-Cos\theta)\right]^{1/2}}\ b<\theta<c \tag{3.29}$$

where

$$Q_1(y)=\frac{d}{dy}\int_b^y \frac{\left(Sin\frac{\theta}{2}\right)^{2\alpha}Q(\theta)Sin\theta d\theta}{(Cos\theta-Cosy)^{1/2}} \tag{3.30}$$

$$\frac{d}{dy}\int_b^y \frac{Sin\theta d\theta}{(Cos\theta-Cosy)^{1/2}(Cost-Cos\theta)^{1/2}}=\frac{Siny(Cost-Cosb)^{1/2}}{(Cosb-Cosy)^{1/2}(Cost-Cosy)^{1/2}} \tag{3.31}$$

in equation (3.29), we get

$$E(y)\int_y^c \frac{h'(u)du}{(Cosy-Cosu)^{1/2}}=Q_1(y)-\frac{1}{\pi}\frac{Siny}{(Cosb-Cosy)^{1/2}}$$

$$\Bigg[\int_0^b \frac{E(t)(Cost-Cosb)^{1/2}dt}{(Cost-Cosy)}\int_b^c \frac{h'(u)du}{(Cost-Cosu)^{1/2}}$$

$$+\int_0^a \frac{E(t)(Cost-Cosb)^{1/2}dt}{(Cost-Cosy)}\int_t^a \frac{g'(u)du}{(Cost-Cosu)^{1/2}}$$

$$+\int_0^\theta \frac{E(t)(Cost-Cosb)^{1/2}dt}{(Cost-Cosy)}\int_d^\pi \frac{k'(u)du}{(Cost-Cosu)^{1/2}}\Bigg]\ b<y<c \tag{3.32}$$

Putting the value of last integral of second term on the right hand side of equation (3.32) from equation (3.18), we get

$$E(y)\int_y^c \frac{h'(u)du}{(Cosb-Cosy)^{1/2}}=Q_1(y)-\frac{1}{\pi}\Bigg[\frac{Siny}{(Cosb-Cosy)^{1/2}}$$

$$\int_0^b \frac{E(t)\left(Cost-Cosb\right)^{1/2}}{\left(Cost-Cosy\right)}dt\int_b^c \frac{h'(u)du}{\left(Cost-Cosu\right)^{1/2}}$$

$$+\int_0^a \frac{E(t)\left(Cost-Cosb\right)^{1/2}}{\left(Cost-Cosy\right)}dt\left\{\frac{P_1(t)}{E(t)}-\int_b^c \frac{h'(u)du}{\left(Cost-Cosu\right)^{1/2}}\right.$$

$$-\int_d^\pi \frac{k'(u)du}{\left(Cost-Cosu\right)^{1/2}}+\int_0^\theta \frac{E(t)\left(Cost-Cosb\right)^{1/2}}{\left(Cost-Cosy\right)}dt$$

$$\left.\times\int_d^\pi \frac{k'(u)du}{\left(Cosy-Cosu\right)^{1/2}}\right]b<y<c \tag{3.33}$$

$$E(y)\int_y^c \frac{h'(u)du}{\left(Cosy-Cosu\right)^{1/2}}=Q_1(y)+P_3(y)-\frac{1}{\pi}\frac{Siny}{\left(Cosb-Cosy\right)^{1/2}}$$

$$\left[\int_a^b \frac{E(t)\left(Cost-Cosb\right)^{1/2}dt}{\left(Cost-Cosy\right)}\int_b^c \frac{h'(u)du}{\left(Cost-Cosu\right)^{1/2}}\right.$$

$$\left.+\int_a^\theta \frac{E(t)\left(Cost-Cosb\right)^{1/2}dt}{\left(Cost-Cosy\right)}\int_d^\pi \frac{k'(u)du}{\left(Cost-Cosu\right)^{1/2}}\right]b<y<c \tag{3.34}$$

where

$$P_3(y)=\frac{Siny}{\left(Cosb-Cosy\right)^{1/2}}\int_b^a \frac{P_1(t)\left(Cost-Cosb\right)^{1/2}dt}{\left(Cost-Cosy\right)} \tag{3.35}$$

Let
$$B(u,\ y)=\frac{Siny}{\left(Cosb-Cosy\right)^{1/2}}\int_a^b \frac{E(t)\left(Cost-Cosb\right)^{1/2}dt}{\left(Cost-Cosy\right)} \tag{3.36}$$

and
$$C(u,\ y)=\frac{Siny}{\left(Cosb-Cosy\right)^{1/2}}\int_a^\theta \frac{E(t)\left(Cost-Cosb\right)^{1/2}dt}{\left(Cost-Cosy\right)} \tag{3.37}$$

Now equation (3.34) can be reduced to the following form

$$E(y)\int_y^c \frac{h'(u)du}{(\mathrm{Cos}t-\mathrm{Cos}u)^{1/2}} = Q_1(y)+P_3(y)$$

$$-\frac{1}{\pi}\left[\int_b^c \frac{h'(u)B(u,\ y)du}{(\mathrm{Cos}t-\mathrm{Cos}u)^{1/2}} + \int_d^\pi \frac{k'(u)C(u,\ y)du}{(\mathrm{Cos}t-\mathrm{Cos}u)^{1/2}}\right] \quad b<y<c \qquad (3.38)$$

Again applying the results (2.5) and (2.7), in equation (3.38) we get

$$E(y)h'(u) = -\frac{1}{\pi}\frac{d}{du}\int_u^c \frac{\{Q_1(y)+P_3(y)\}\mathrm{Sin}s ds}{(\mathrm{Cos}u-\mathrm{Cos}s)^{1/2}} + \frac{1}{\pi^2}\frac{d}{du}\int_u^c \frac{\mathrm{Sin}s ds}{(\mathrm{Cos}u-\mathrm{Cos}s)^{1/2}}$$

$$\left[\int_b^c \frac{B(\nu,\ y)h'(\nu)d\nu}{(\mathrm{Cos}s-\mathrm{Cos}\nu)^{1/2}} + \int_d^\pi \frac{C(\nu,\ y)k'(\nu)d\nu}{(\mathrm{Cos}s-\mathrm{Cos}\nu)^{1/2}}\right] \quad b<y<c \qquad (3.39)$$

Changing the order of integration and using the result

$$\frac{d}{du}\int_c^u \frac{\mathrm{Sin}s ds}{(\mathrm{Cos}u-\mathrm{Cos}s)^{1/2}(\mathrm{Cos}s-\mathrm{Cos}\nu)^{1/2}}$$

$$= \frac{\mathrm{Sin}s ds(\mathrm{Cos}\nu-\mathrm{Cos}c)^{1/2}}{(\mathrm{Cos}c-\mathrm{Cos}s)^{1/2}(\mathrm{Cos}\nu-\mathrm{Cos}s)^{1/2}} \qquad (3.40)$$

Equation (3.39) can be written as

$$E(y)h'(u) = Q_2(u) + \frac{1}{\pi^2}\int_b^c h'(\nu)B(s,\ \nu)d\nu + \frac{1}{\pi^2}\int_d^\pi k'(\nu)C(s,\ \nu)d\nu$$

$$b<u<c \qquad (3.41)$$

where

$$Q_2(y) = -\frac{1}{\pi}\frac{d}{du}\int_u^c \frac{\{Q_1(y)+P_3(y)\}\mathrm{Sin}s ds}{(\mathrm{Cos}u-\mathrm{Cos}s)^{1/2}} \qquad (3.42)$$

$$B(s,\ \nu) = \frac{\mathrm{Sin}u(\mathrm{Cos}\nu-\mathrm{Cos}c)^{1/2}}{(\mathrm{Cos}c-\mathrm{Cos}s)^{1/2}(\mathrm{Cos}\nu-\mathrm{Cos}s)^{1/2}}B(\nu,\ y) \qquad (3.43)$$

and $$C(s,\ \nu) = \frac{\mathrm{Sin}u(\mathrm{Cos}\nu-\mathrm{Cos}c)^{1/2}}{(\mathrm{Cos}c-\mathrm{Cos}s)^{1/2}(\mathrm{Cos}\nu-\mathrm{Cos}s)^{1/2}}C(\nu,\ y)$$

(3.44)

Again using summation result (2.3) in terms of integral in equation (3.10) we get

$$\left[\int_d^\theta k'(u) + \int_\theta^\pi k'(u)\right]\int_0^{\min(u,\,\theta)} \frac{E(y)}{\left[(\mathrm{Cosy}-\mathrm{Cosu})(\mathrm{Cosy}-\mathrm{Cos}\theta)\right]^{1/2}}\,\mathrm{dudy}$$

$$= \pi\left(\mathrm{Sin}\frac{\theta}{2}\right)^{2\alpha} R(\theta) - \left[\int_0^a g'(u)du + \int_b^c h'(u)du\right]$$

$$\times\int_0^{\min(u,\,\theta)} \frac{E(y)dy}{(\mathrm{Cosy}-\mathrm{Cosu})(\mathrm{Cosy}-\mathrm{Cos}\theta)^{1/2}} \qquad d<\theta<c \qquad (3.45)$$

$$\int_d^\theta k'(u)du\int_0^u \frac{E(y)dy}{(\mathrm{Cosy}-\mathrm{Cosu})^{1/2}(\mathrm{Cosy}-\mathrm{Cos}\theta)^{1/2}}$$

or

$$+\int_\theta^\pi k'(u)du\int_0^\theta \frac{E(y)dy}{(\mathrm{Cosy}-\mathrm{Cosu})^{1/2}(\mathrm{Cosy}-\mathrm{Cos}\theta)^{1/2}}$$

$$= \pi\left(\mathrm{Sin}\frac{\theta}{2}\right)^{2\alpha} R(\theta) - \left[\int_0^a g'(u)du + \int_b^c h'(u)du\right]$$

$$\int_0^\theta \frac{E(y)dy}{(\mathrm{Cosy}-\mathrm{Cosu})^{1/2}(\mathrm{Cosy}-\mathrm{Cos}\theta)^{1/2}} \qquad d<\theta<\pi \qquad (3.46)$$

Changing the order of integration in equation (3.46), we obtain

$$\int_d^\theta \frac{E(y)dy}{(\mathrm{Cosy}-\mathrm{Cos}\theta)^{1/2}}\int_y^\pi \frac{k'(u)du}{(\mathrm{Cosy}-\mathrm{Cosu})^{1/2}} = \pi\left(\mathrm{Sin}\frac{\theta}{2}\right)^{2\alpha} R(\theta)$$

$$-\left[\int_0^a \frac{E(y)dy}{(\mathrm{Cosy}-\mathrm{Cos}\theta)^{1/2}}\int_0^a \frac{g'(u)du}{(\mathrm{Cosy}-\mathrm{Cosu})^{1/2}} + \int_0^b \frac{E(y)dy}{(\mathrm{Cosy}-\mathrm{Cos}\theta)^{1/2}}\right.$$

$$\times\int_b^c \frac{h'(u)du}{(\mathrm{Cosy}-\mathrm{Cosu})^{1/2}} + \int_b^c \frac{E(y)dy}{(\mathrm{Cosy}-\mathrm{Cos}\theta)^{1/2}}\int_y^c \frac{h'(u)du}{(\mathrm{Cosy}-\mathrm{Cosu})^{1/2}}$$

$$\left.+\int_0^d \frac{E(y)dy}{(\mathrm{Cosy}-\mathrm{Cos}\theta)^{1/2}}\int_d^\pi \frac{k'(u)du}{(\mathrm{Cosy}-\mathrm{Cosu})^{1/2}}\right] \qquad d<\theta<\pi \qquad (3.47)$$

Using the results (2.4) and (2.6), we get

$$E(y)\int_y^{\pi}\frac{k'(u)du}{(Cosy-Cosu)^{1/2}}=\frac{d}{dy}\int_d^y\frac{\left(Sin\frac{\theta}{2}\right)^{2\alpha}.R(\theta)Sin\theta d\theta}{(Cos\theta-Cosy)^{1/2}}$$

$$-\frac{1}{\pi}\frac{d}{dy}\int_d^y\frac{Sin\theta d\theta}{(Cos\theta-Cosy)^{1/2}}\left[\int_0^a\frac{E(y)dy}{(Cosy-Cos\theta)^{1/2}}\int_y^a\frac{g'(u)du}{(Cosy-Cosu)^{1/2}}\right.$$

$$+\int_0^b\frac{E(y)dy}{(Cosy-Cos\theta)^{1/2}}\int_b^c\frac{h'(u)du}{(Cosy-Cosu)^{1/2}}+\int_b^c\frac{E(y)dy}{(Cosy-Cos\theta)^{1/2}}$$

$$\left.\times\int_y^c\frac{h'(u)du}{(Cosy-Cosu)^{1/2}}+\int_0^d\frac{E(y)dy}{(Cosy-Cos\theta)^{1/2}}\int_d^{\pi}\frac{k'(u)du}{(Cosy-Cosu)^{1/2}}\right]$$

$$d<y<\pi \qquad (3.48)$$

Changing the order of integration and applying the result.

$$\frac{d}{dy}\int_d^y\frac{Sin\theta d\theta}{(Cos\theta-Cosy)^{1/2}(Cost-Cos\theta)^{1/2}}=\frac{Siny(Cost-Cosd)^{1/2}}{(Cosd-Cosy)^{1/2}(Cost-Cosy)^{1/2}} \qquad (3.49)$$

in equation (3.48), we get

$$E(y)\int_y^{\pi}\frac{k'(u)du}{(Cosy-Cosu)^{1/2}}=R_1(y)-\frac{1}{\pi}\frac{Siny}{(Cosd-Cosy)^{1/2}}$$

$$\times\left[\int_0^a\frac{E(t)(Cost-Cosd)^{1/2}dt}{(Cost-Cosy)}\int_t^a\frac{g'(u)du}{(Cosy-Cosu)^{1/2}}\right.$$

$$+\int_0^b\frac{E(t)(Cost-Cosd)^{1/2}dt}{(Cost-Cosy)^{1/2}}\int_b^c\frac{h'(u)du}{(Cost-Cosu)^{1/2}}$$

$$+\int_t^c\frac{E(t)(Cost-Cosd)^{1/2}dt}{(Cost-Cosy)^{1/2}}\frac{h'(u)du}{(Cost-Cosu)^{1/2}}$$

$$+\int_0^d \frac{E(t)\left(\text{Cost}-\text{Cosd}\right)^{1/2}dt}{\left(\text{Cosy}-\text{Cosy}\right)^{1/2}}\int_d^{\pi}\frac{k'(u)du}{\left(\text{Cost}-\text{Cosu}\right)^{1/2}}\Bigg] \quad d<y<\pi \tag{3.50}$$

where

$$R_1(y)=\frac{d}{dy}\int_d^y \frac{\left(\text{Sin}\frac{\theta}{2}\right)^{2\alpha}.R(\theta)\text{Sin}\theta d\theta}{\left(\text{Cos}\theta-\text{Cosy}\right)^{1/2}} \tag{3.51}$$

Now putting the values of last integral of first and third terms on the right hand side from equations (3.18) and (3.38), we get

$$E(y)\int_y^{\pi}\frac{k'(u)du}{\left(\text{Cosy}-\text{Cosu}\right)^{1/2}}=R_1(y)+R_2(y)$$

$$-\frac{1}{\pi}\left[\int_b^c \frac{h'(u)U(u,\ y)du}{\left(\text{Cost}-\text{Cosu}\right)^{1/2}}+\int_d^{\pi}\frac{k'(u)V(u,\ y)du}{\left(\text{Cost}-\text{Cosu}\right)^{1/2}}\right] \quad d<y<\pi \tag{3.52}$$

where

$$R_2(y)=\frac{\text{Siny}}{\left(\text{Cosd}-\text{Cosy}\right)^{1/2}}\left[\int_b^a \frac{P_1(t)\left(\text{Cost}-\text{Cosd}\right)^{1/2}}{\left(\text{Cost}-\text{Cosy}\right)}dt\right.$$

$$\left.+\int_b^c \frac{\left\{Q_1(t)+P_3(t)\right\}\left(\text{Cost}-\text{Cosd}\right)^{1/2}dt}{\left(\text{Cost}-\text{Cosy}\right)}\right] \tag{3.53}$$

$$U(v,\ y)=\frac{\text{Siny}}{\left(\text{Cosd}-\text{Cosy}\right)^{1/2}}\left[\int_0^a \frac{E(t)\left(\text{Cost}-\text{Cosd}\right)^{1/2}}{\left(\text{Cost}-\text{Cosy}\right)}dt\right.$$

$$\left.+\int_0^b \frac{E(t)\left(\text{Cost}-\text{Cosd}\right)^{1/2}dt}{\left(\text{Cost}-\text{Cosy}\right)}+\int_b^c \frac{E(t)B(v,\ t)(\text{Cost}-\text{Cosd})^{1/2}dt}{\left(\text{Cost}-\text{Cosy}\right)}\right] \tag{3.54}$$

$$V(v,\ y)=\frac{\text{Siny}}{\left(\text{Cosd}-\text{Cosy}\right)^{1/2}}\left[\int_0^a \frac{E(t)\left(\text{Cost}-\text{Cosd}\right)^{1/2}dt}{\left(\text{Cost}-\text{Cosy}\right)}\right.$$

$$+\int_0^b \frac{E(t)\left(\mathrm{Cos}t-\mathrm{Cos}d\right)^{1/2}dt}{\left(\mathrm{Cos}t-\mathrm{Cos}y\right)}+\int_b^c \frac{E(t)C(v,\ t)(\mathrm{Cos}t-\mathrm{Cos}d)^{1/2}dt}{\left(\mathrm{Cos}t-\mathrm{Cos}y\right)}\Bigg] \tag{3.55}$$

Again applying the results (2.5) and (2.7) in equation (3.52), we get

$$E(y)h'(u)=-\frac{1}{\pi}\frac{d}{du}\int_u^\pi \frac{\{R_1(y)+R_2(y)\}\mathrm{Sin}s ds}{\left(\mathrm{Cos}u-\mathrm{Cos}s\right)^{1/2}}$$

$$+\frac{1}{\pi^2}\frac{d}{du}\int_u^\pi \frac{\mathrm{Sin}s ds}{(\mathrm{Cos}u-\mathrm{Cos}s)^{1/2}}\left[\int_b^c \frac{U(\nu,\ y)h'(\nu)d\nu}{(\mathrm{Cos}s-\mathrm{Cos}\nu)^{1/2}}+\int_d^\pi \frac{V(\nu,\ y)k'(\nu)d\nu}{(\mathrm{Cos}s-\mathrm{Cos}\nu)^{1/2}}\right] \tag{3.56}$$

Changing the order of integration and using the result

$$\frac{d}{du}\int_\pi^\alpha \frac{\mathrm{Sin}s ds}{\left(\mathrm{Cos}u-\mathrm{Cos}s\right)^{1/2}\left(\mathrm{Cos}s-\mathrm{Cos}\nu\right)^{1/2}}=\frac{\mathrm{Sin}u\left(\mathrm{Cos}\nu-\mathrm{Cos}\pi\right)^{1/2}}{\left(\mathrm{Cos}\pi-\mathrm{Cos}s\right)^{1/2}\left(\mathrm{Cos}\nu-\mathrm{Cos}s\right)} \tag{3.57}$$

in equation (3.56), we get

$$E(y)k'(u)=R_3(u)+\frac{1}{\pi^2}\int_b^c h'(\nu)U(s,\ \nu)d\nu+\frac{1}{\pi^2}\int_d^\pi k'(\nu)V(s,\ \nu)d\nu$$

$$d<u<\pi \tag{3.58}$$

where

$$R_3(u)=-\frac{1}{\pi}\frac{d}{du}\int_u^\pi \frac{\{R_1(y)+R_2(y)\}\mathrm{Sin}s ds}{\left(\mathrm{Cos}u-\mathrm{Cos}s\right)^{1/2}} \tag{3.59}$$

$$U(s,\ \nu)=\frac{\mathrm{Sin}u\left(\mathrm{Cos}\nu-\mathrm{Cos}\pi\right)^{1/2}}{\left(\mathrm{Cos}\pi-\mathrm{Cos}s\right)^{1/2}\left(\mathrm{Cos}\nu-\mathrm{Cos}s\right)}U(\nu,\ \nu) \tag{3.60}$$

and

$$V(s,\ \nu)=\frac{\mathrm{Sin}u\left(\mathrm{Cos}\nu-\mathrm{Cos}\pi\right)^{1/2}}{\left(\mathrm{Cos}\pi-\mathrm{Cos}s\right)^{1/2}\left(\mathrm{Cos}\nu-\mathrm{Cos}s\right)}V(\nu,\ \nu) \tag{3.61}$$

Equations (3.22), (3.41) and (3.58) are Fredholm integral equations of the second kind which determine $g'(u)$, $h'(u)$ $k'(u)$ respectively. Knowing the values of $g'(u)$, $h'(u)$ and $k'(u)$, the values of coefficients A_n, can be obtained from (3.4).

Particular Cases

If we let $d=\pi$ in equations (1.1) to (1.5), they reduce to the corresponding quadruple series equations considered earlier in [2] and the above solution agrees with that obtained previously. Similarly, the solution of corresponding triple and dual series can be obtained as particular cases.

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