ISSN: 0975-833X

Available online at http://www.journalcra.com

International Journal of Current Research
Vol. 7, Issue, 06, pp.16789-16794, June, 2015

INTERNATIONAL JOURNAL OF CURRENT RESEARCH

RESEARCH ARTICLE

A NON-SYMMETRIC DIVERGENCE AND KULLBACK-LEIBLER DIVERGENCE MEASURE

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ARTICLE INFO

Article History:
Received 20th March, 2015
Received in revised form
08th April, 2015
Accepted 24th May, 2015
Published online 27th June, 2015

Key words:
Csiszar's f-Divergence Measure,
Kullback-Leibler Divergence Measure,
Information Inequalities etc.

ABSTRACT

Information divergence measures and their bounds are well known in the literature of Information Theory. In this research article, we shall consider a new non-symmetric information divergence measure. Upper and lower bounds of non-symmetric divergence measure in terms of Kullback-Leibler divergence measure have been studied. Numerical bounds of new divergence measures are also discussed.

***Citation*:** Ram Naresh Saraswat, 2015, "A non-symmetric divergence and kullback-leibler divergence measure", *International Journal of Current Research*, 7, (6), 16789-16794.

INTRODUCTION

Let

$$\Gamma_n = \left\{ P = (p_1, p_2, \ldots\ldots. p_n) \middle| p_i \geq 0, \sum_{i=1}^{n} p_i = 1 \right\}, n \geq 2 \quad \ldots\ldots (1.1)$$

be the set of all complete finite discrete probability distributions. There are many information and divergence measures are exist in the literature on Information Theory. Csiszar (Csiszar, 1961) and (Csiszar, 1978) introduced a generalized measure of information using f-divergence measure is given by

$$I_f(P,Q) = \sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right) \quad \ldots\ldots (1.2)$$

where $f : \mathbf{R}_+ \to \mathbf{R}_+$ is a convex function and $P, Q \in \Gamma_n$

Here we list existing divergence measure which is the category of Csiszar's f-divergence measure, together with the suitable generating function f.

Kullback-Leibler divergence measure (Kullback and Leibler, 1951)

(i) If $f(t) = -\log t$ then kullback and Leibler divergence measure is given by

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$$I_f(P,Q) = D(Q,P) = \sum_{i=1}^{n} q_i \log \frac{q_i}{p_i} \qquad \text{(1.3)}$$

(ii) If $f(t) = t \log t$ then kullback and Leibler divergence measure is given by

$$I_f(P,Q) = D(P,Q) = \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i} \qquad \text{(1.4)}$$

In whole paper, in the section 2, we have introduced information inequalities. New non-symmetric information divergence measure has derived in section 3. Bounds of new information divergence measure in terms of Kullback-Leibler divergence measure have studied in section 4. In section 5, give the numerical bounds of new non-symmetric information divergence in terms of Kullback-Leibler divergence measure.

INEQUALITIES RELATED WITH CSISZAR'S F-DIVERGENCE MEASURES

The following proposition is one of the results of the theorem given in (Taneja and Kumar, 2004) and similar line to (Dragomir, 2001; Jain and Saraswat, 2013 and Jain and Saraswat, 2013).

Proposition 2.1:- Let $f:(0,\infty) \to \mathbf{R}$ be a mapping which is normalized i.e. $f(1)=0$ and satisfies the assumptions.

(i) f is twice differentiable on (r, R) .where $0 \le r \le 1 \le R \le \infty$

(ii) there exist the real Constants m , M such that $m < M$

$$m \le t^{2-s} f''(t) \le M, \forall t \in (r,R), s \in \mathbf{R} \qquad \text{(2.1)}$$

If $P, Q \in \Gamma_n$ are discrete probability distributions satisfying assumption

$$0 < r \le \frac{p_i}{q_i} \le R < \infty, \ \forall i \in \{1,2,3,.........n\} \qquad \text{(2.1)}$$

then we have the inequality

$$m\ \Phi_s(P,Q) \le I_f(P,Q) \le M\ \Phi_s(P,Q) \ \text{(2.3)}$$

The case s=0, s=1 of proposition (2.1) gives

Proposition 2.2:-Let $f:(0,\infty) \to \mathbf{R}$ is normalized i.e. $f(1)=0$ and satisfies the assumptions.

(i) f is twice differentiable on (r, R) .where $0 \le r \le 1 \le R \le \infty$

(ii) there exist Constant m , M such that $m < M$

$$m \le t^2 f''(t) \le M, \ \forall \ t \in (r,R) \qquad \text{(2.4)}$$

$$m \le t f''(t) \le M, \ \forall \ t \in (r,R) \qquad \text{(2.5)}$$

If $P, Q \in \Gamma_n$ are discrete probability distributions satisfying assumption

$$0 < r \le \frac{p_i}{q_i} \le R < \infty, \ \forall \ i \in \{1,2,3,.........n\}$$

then we have the inequality corresponding to s=0 & s=1

$$m\ D(Q,P) \le I_f(P,Q) \le M\ D(Q,P) \qquad \text{(2.6)}$$

$$m\ D(P,Q) \le I_f(P,Q) \le M\ D(P,Q) \qquad \text{(2.7)}$$

In view of proposition (4.1) we can state the following results.

NON-SYMMETRIC DIVERGENCE MEASURE

In this section we introduce a new information divergence measure which is the category of Csiszar's f-divergence measure. Let us consider the function $f:(0,\infty)\to \mathbf{R}$

$$f(t)=\frac{(t^2-1)^2}{t},\ f'(t)=\frac{3t^4-2t^2-1}{t^2},\ f''(t)=\frac{6t^4+2}{t^3}>0,\forall t>0 \quad \text{.......} \quad (3.1)$$

Hence function f (t) is convex from equation (3.1) and figure 3.1, and $f(1)=0$ i.e. normalized.

The f-divergence measure corresponding to function (3.1) is given by

$$I_f(P,Q)=\sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right)=\sum_{i=1}^{n}\frac{(p_i^2-q_i^2)^2}{q_i^2 p_i}=4\sum_{i=1}^{n}\left(1\Big/\frac{2p_iq_i}{p_i+q_i}\right)\frac{(p_i+q_i)}{2}\frac{(p_i-q_i)^2}{q_i}=N(P,Q) \quad \text{.......} \quad (3.2)$$

where "$N(P,Q)$" is may be combination of Harmonic , Arithmetic and χ^2 -divergence measure.

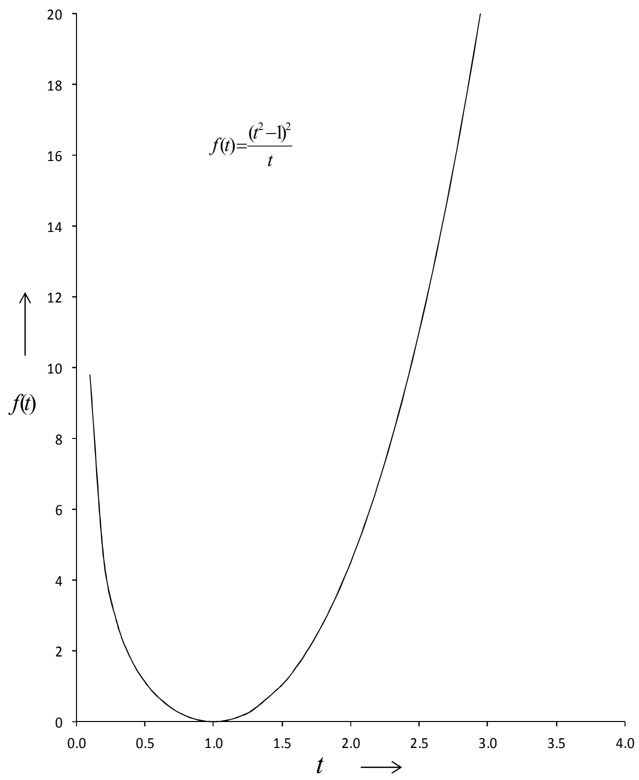

Figure 3.1. Graphical presentation of New Measure $N(P,Q)$

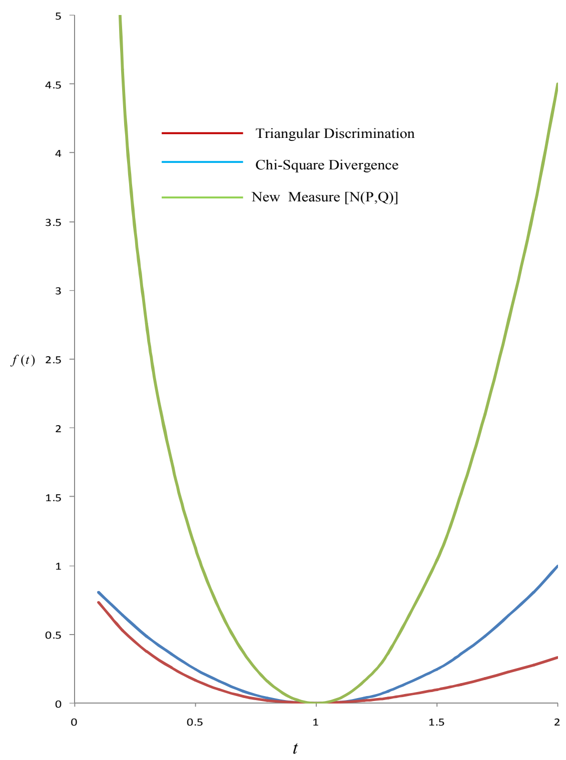

Figure 3.2. Comparison of New Measure $N(P,Q)$ with some well-known divergence measures

It is clear that from the figure 3.1 and 3.2 the convex function $f(t)$ gives a steeper slope. Further $f(1)=0$, so that $N(P,P)=0$ and the convexity of the function $f(t)$ ensure that the measure (3.2) is non-negative.

In following sections 4 we present particular cases of the proposition (2.1) using the measure $N(P,Q)$ given in equation (3.2).

BOUNDS IN TERMS OF KULLBACK-LEIBLER DIVERGENCE MEASURE

RESULTS

Result 4.1.1:- Let $P,Q \in \Gamma_n$ and $s=0$ .Let there exists r, R such that r < R and

$$0 < r \le \frac{p_i}{q_i} \le R < \infty, \forall i \in \{1,2,3,.........n\}$$

(i) If $r \in (0, \frac{1}{\sqrt{3}})$ then

$$\frac{8\sqrt{3}}{3} D(Q,P) \approx 4.61 D(Q,P) \le N(P,Q) \le \max\left\{\frac{6r^4+2}{r}, \frac{6R^4+2}{R}\right\} D(Q,P) \quad \text{(4.1)}$$

(ii) If $r \in (\frac{1}{\sqrt{3}}, \infty)$

$$\frac{6R^4+2}{R} D(Q,P) \le N(P,Q) \le \frac{6r^4+2}{r} D(Q,P) \quad \text{(4.2)}$$

Proof: - From equations (3.1), (3.2) & (4.1), we get

$$g(t) = t^2 f''(t) = \frac{6t^4+2}{t} \quad ,\forall t > 0$$

we have $g'(t) = 18t^2 - \frac{2}{t^2} = 0$,

$g'(t) = 0$ gives $t_0 = \frac{1}{\sqrt{3}} \approx .58$

$$g''(t) = 36t + \frac{4}{t^3},$$

and

$$g''(.58) = 36(.58) + \frac{4}{(.58)^3} = 49.3 \text{ (Positive)}$$

which shows that function g (t) has minimum realized at $t_0 = .58$ and $\min g(t_0) = m$

we have two cases:-

(i)If $0 < r \le \frac{1}{\sqrt{3}}$, then

$$\left.\begin{aligned} m &= \inf_{t\in[r,R]} g(t) = g(t_0) = \frac{8\sqrt{3}}{3} \approx 4.61 \\ M &= \sup_{t\in[r,R]} g(t) = \max\left\{\frac{6r^4+2}{r}, \frac{6R^4+2}{R}\right\} \end{aligned}\right\} \quad \text{(4.3)}$$

(ii) If $\frac{1}{\sqrt{3}} < r < \infty$, then

$$m = \inf_{t\in[r,R]} g(t) = \frac{6r^4+2}{r}, M = \sup_{t\in[r,R]} g(t) = \frac{6R^4+2}{R} \quad \text{(4.4)}$$

Equations (2.6) & (2.7) of proposition (2.2) using equations (3.2), (4.3) and (4.4) gives the result (4.1) & (4.2).

Result 4.1.2:-

Let $P,Q \in \Gamma_n$ and $s = 1$. Let there exists r, R such that r < R and $0 < r \le \frac{p_i}{q_i} \le R < \infty$ $,\forall i \in \{1,2,3,.........n\}$.

(i)If $0 < r \leq 0.76$, **then**

$$4\sqrt{3}D(P,Q) \leq N(P,Q) \leq \max\left[\frac{6r^4+2}{r^2}, \frac{6R^4+2}{R^2}\right] D(P,Q) \quad \ldots (4.5)$$

(ii)If $0.76 < r \leq \infty$ **, then**

$$\frac{6r^4+2}{r^2} D(P,Q) \leq N(P,Q) \leq \frac{6R^4+2}{R^2} D(P,Q) \quad \ldots (4.6)$$

Proof: - From equations (3.1), (3.2) & (4.2), we get

$$g(t) = tf''(t) = \frac{6t^4+2}{t^2} \quad , \forall\, t > 0$$

we have $g'(t) = 12t - \frac{4}{t^3} = 0$,

$g'(t) = 0$ gives $t_0 = \left(\frac{1}{3}\right)^{1/4} \approx 0.76$

$g''(t) = 12 + \frac{12}{t^3}$,

and

$g''(0.76) = \text{(Positive)}$

Which shows that function g (t) has minimum realized at $t_0 = \left(\frac{1}{3}\right)^{1/4} \approx 0.76$.

We have two cases:-

(i) If $0 < r \leq 0.76$, then

$$m = \inf_{t\in[r,R]} g(t) = g(t_0) = 4\sqrt{3} \quad \ldots (4.7)$$

$$M = \sup_{t\in[r,R]} = \max\left[\frac{6r^4+2}{r^2}, \frac{6R^4+2}{R^2}\right] \quad \ldots (4.8)$$

(ii) If $0.76 < r < \infty$, then

$$m = \inf_{t\in[r,R]} g(t) = \frac{6r^4+2}{r^2},\; M = \sup_{t\in[r,R]} = \frac{6R^4+2}{R^2} \quad \ldots (4.9)$$

Equations (2.6) and (2.7) of proposition (2.2) using equation (3.2), (4.7), (4.8) & (4.9) give the results (4.5) & (4.6).

NUMERICAL ILLUSTRATIONS

Let P be the binomial probability distribution for the random valuable X with parameter (n=8 p=0.5) and Q its approximated normal probability distribution.The following table have also discussed by Pranesh Kumar and Andrew Johnson in 2005.

Table 5.1

x	0	1	2	3	4	5	6	7	8
p (x)	0.004	0.031	0.109	0.219	0.274	0.219	0.109	0.031	0.004
q (x)	0.005	0.030	0.104	0.220	0.282	0.220	0.104	0.030	0.005
p(x)/q(x)	0.774	1.042	1.0503	0.997	0.968	0.997	1.0503	1.042	0.774
N(P, Q)	0.00081	0.00013	0.00096	0.00018	0.0009	0.00018	0.00096	0.00013	0.0008

Here r = 0.77 and R = 1.05 are the lower and upper bounds. Now, we shall discuss the numerical bounds of new non-symmetric information divergence measure in terms of Kullback-Leibler divergence measure using equation (4.1), (4.2), (4.5) and (4.6) and the above table, then we get,

(i) If $r \in (0, \frac{1}{\sqrt{3}})$ **then**

$$\frac{8\sqrt{3}}{3} D(Q,P) \approx 4.61 D(Q,P) \le N(P,Q) \le \max\left\{\frac{6r^4+2}{r}, \frac{6R^4+2}{R}\right\} D(Q,P)$$

$$\frac{8\sqrt{3}}{3} D(Q,P) \approx 4.61 D(Q,P) \le N(P,Q) \le \max\left\{\frac{6(.77)^4+2}{(.77)}, \frac{6(1.05)^4+2}{(1.05)}\right\} D(Q,P)$$

$$\frac{8\sqrt{3}}{3} D(Q,P) \approx 4.61 D(Q,P) \le N(P,Q) \le \max\left\{5.336, 8.850\right\} D(Q,P)$$

$$4.61 D(Q,P) \le N(P,Q) \le [8.850] D(Q,P)$$

(ii) If $r \in (\frac{1}{\sqrt{3}}, \infty)$

$$\frac{6R^4+2}{R} D(Q,P) \le N(P,Q) \le \frac{6r^4+2}{r} D(Q,P)$$

$$[8.850] D(Q,P) \le N(P,Q) \le [5.336]\ D(Q,P)$$

(iii)If $0 < r \le 0.76$**, then**

$$4\sqrt{3} D(P,Q) \le N(P,Q) \le \max\left[\frac{6r^4+2}{r^2}, \frac{6R^4+2}{R^2}\right] D(P,Q)$$

$$4\sqrt{3} D(P,Q) \le N(P,Q) \le \max\left[\frac{6(.77)^4+2}{(.77)^2}, \frac{6(1.05)^4+2}{(1.05)^2}\right] D(P,Q)$$

$$4\sqrt{3} D(P,Q) \le N(P,Q) \le \max\left[6.930, 8.4290\right] D(P,Q)$$

$$4\sqrt{3} D(P,Q) \le N(P,Q) \le [8.4290]\ D(P,Q)$$

(iv)If $0.76 < r \le \infty$ **, then**

$$\frac{6r^4+2}{r^2} D(P,Q) \le N(P,Q) \le \frac{6R^4+2}{R^2} D(P,Q)$$

$$[6.930] D(P,Q) \le N(P,Q) \le [8.4290] D(P,Q)$$

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