Available online at http://www.journalcra.com

International Journal of Current Research
Vol. 7, Issue, 07, pp.18456-18461, July, 2015

INTERNATIONAL JOURNAL OF CURRENT RESEARCH

ISSN: 0975-833X

RESEARCH ARTICLE

INTEGRAL AND COMPUTATIONAL REPRESENTATIONS OF GENERALIZED HURWITZ-LERCH ZETA FUNCTION $\Phi(a_1 z, s, a)$

*Harish Nagar and Alka Tank

Department of Mathematics, Mewar University, Gangrar, Chittorgarh (Raj), India

ARTICLE INFO

Article History:
Received 23rd April, 2015
Received in revised form
19th May, 2015
Accepted 15th June, 2015
Published online 31st July, 2015

Key words:
Generalized Riemann Zeta function,
Integral representations of special functions,
Generalized Hurwitz-Lerch Zeta Function,
Gamma function.

ABSTRACT

This article presents a systematic investigation of various integral and computational representations for family of General Hurwitz – Lerch Zeta function $\Phi(z,s,a)$. In this paper new results for Generalized Hurwitz – Lerch Zeta function $\Phi(a_1 z,s,a)$are established.Certain known integral representation for generalized Riemann Zeta function is obtained as special cases of main results. In this paper we established five results, these results presents the different forms of integral representations of generalized Hurwitz-Lerch Zeta function.

Citation: Harish Nagar and Alka Tank, 2015. "Integral and Computational representations of Generalized Hurwitz-Lerch Zeta Function $\varphi(a_1 z,s,a)$", *International Journal of Current Research*, 7, (7), 18456-18461.

INTRODUCTION

Definitions:

(i)The generalized Hurwitz-Lerch Zeta function is defined by [2, p. 100, eq.(1.5)]; (see, eq. [1, p.27, eq. 1.11 (1)]; see also [5, p.121] and [6, p.194]).

$$\Phi(z,s,a) = \sum_{n=0}^{\infty} \frac{(z)^n}{(a+n)^s} \quad \ldots (1.1)$$

$$\text{or } \Phi(a_1 z,s,a) = \sum_{n=0}^{\infty} \frac{(a_1 z)^n}{(a+n)^s} \quad \ldots (1.2)$$

$|z| < 1, Re\ (a) > 0,\ Re\ (a_1) > 0$

(ii)Integral representation of the Generalized Hurwitz-Lerch Zeta function $\Phi(z,s,a)$(see, eq. [1, p. 27, eq. 1.11 (3)]; [2, p. 100, eq. (1.6)]; see also [6, p.194, eq. 2.5 (4)] is given by

$$\Phi(z,s,a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-at}}{1-ze^{-t}} dt \quad \ldots (1.3)$$

$$\text{or } \Phi(a_1 z,s,a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-at}}{1-a_1 ze^{-t}} dt \quad \ldots (1.4)$$

Corresponding author: Harish Nagar,
Department of mathematics, Mewar University, Gangrar, Chittorgarh (Raj), India.

(iii)The generalized Hurwitz-Lerch Zeta function defined in (1.1) reduce to Riemannian Zeta function $\zeta(s)$ for z = 1, the Hurwitz (or generalized) Zeta function $\zeta(s,a)$ and Lerch Zeta function $\ell_s(\xi)$ defined by (see, for details, [1, Chapter I]; [5, Chapter 2]; and [1, p. 27, eq. (1)].

$$\zeta(s) = \sum_{n=0}^{\infty} \frac{1}{(1+n)^s} = \Phi(1,s,1);\ Re(s) > 1 \qquad (1.5)$$

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(a+n)^s} = \Phi(1,s,a);(Re(s) > 1;\ a \in \mathbb{C}\backslash Z_0^-) \qquad (1.6)$$

and $$\ell_s(\xi) = \sum_{n=0}^{\infty} \frac{e^{2n\pi i\xi}}{(1+n)^s} = \Phi\left(e^{2\pi i\xi}, s, 1\right);\ (Re(s) > 1;\ \xi \in \mathbb{R}), \qquad (1.7)$$

(iv)The integral representation of Riemannian Zeta function [1, p. 32, eq. (1)].

$$\zeta(s,a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-at}}{1-e^{-t}} dt \qquad (1.8)$$

$Re(s) > 1, \quad (a) > 0$

(v)The Gamma function defined by [3, p. 884, eq. 8.312 (2)]

$$\int_0^{\infty} e^{-px} x^{n-1} dx = \frac{\Gamma(n)}{p^n} \qquad (1.9)$$

Results Required

The following results are required here
(i)The Gamma function is defined by [3, p. 884, eq. 8.312 (2)]

$$\Gamma(n) = a^n \int_0^{\infty} x^{n-1} e^{-ax} dx, \quad Re(x) > 0, n > 0 \qquad (1.10)$$

(ii)The sum of finite and infinite geometric progression are also required

(see, [3, p. 1, eq. (0.112); p. 8, eq. (0.231) (1)].

$$\sum_{k=1}^{n} aq^{k-1} = \frac{a(q^n - 1)}{q-1} \qquad (1.11)$$
$[q \neq 1]$

$$\sum_{k=0}^{n} aq^k = \frac{a}{1-q},\ |q| < 1 \qquad (1.12)$$

Main Results

We have established here the following five results, these results presents the different forms of integral representations of Generalized Hurwitz – Lerch Zeta function.
Result – 1:

$$\Phi(a_1 z, s, a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-at}}{1-a_1 z e^{-t}} \left(1 - (a_1 z)^{(r+1)} e^{-(r+1)t}\right) dt + \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-at}}{1-a_1 z e^{-t}} (a_1 z)^{r+1} e^{-(r+1)} dt \qquad (2.1)$$
$Re(s) > 1, |z| < 1, Re(a) > 0, Re\ (a_1) > 0$

In particular

$$\Phi(a_1 z, -s, a) = \frac{1}{\Gamma(-s)} \int_0^{\infty} \frac{t^{-s-1} e^{-at}}{1-a_1 z e^{-t}} \left(1 - (a_1 z)^{(r+1)} e^{-(r+1)t}\right) dt + \frac{1}{\Gamma(-s)} \int_0^{\infty} \frac{t^{-s-1} e^{-at}}{1-a_1 z e^{-t}} (a_1 z)^{r+1} e^{-(r+1)} dt \qquad (2.2)$$

Result – 2:

$$\Phi(a_1 z, s, a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-at}}{1-a_1 z e^{-t}} \left(1 - (a_1 z)^{(r+1)} e^{-(r+1)t}\right) dt + \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-at}}{1-a_1 z e^{-t}} (a_1 z)^{r+1} e^{-(r+1)} (1 - (a_1 z)^k e^{-kt}) dt + \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-at}}{1-a_1 z e^{-t}} (a_1 z)^{r+k+1} e^{-(r+k+1)} dt \qquad (2.3)$$

Provided that, $Re(s) > 1, |z| < 1, Re(a) > 0, r \in Z_0^+, k \in Z^+$
In particular

$$\Phi(a_1 z, -s, a) = \frac{1}{\Gamma(-s)} \int_0^\infty \frac{t^{-s-1} e^{-at}}{1 - a_1 z e^{-t}} \left(1 - (a_1 z)^{(r+1)} e^{-(r+1)t}\right) dt + \frac{1}{\Gamma(-s)} \int_0^\infty \frac{t^{-s-1} e^{-at}}{1 - a_1 z e^{-t}} (a_1 z)^{r+1} e^{-(r+1)} (1 - (a_1 z)^k e^{-kt}) dt + \frac{1}{\Gamma(-s)} \int_0^\infty \frac{t^{-s-1} e^{-at}}{1 - a_1 z e^{-t}} (a_1 z)^{r+k+1} e^{-(r+k+1)} dt \quad \ldots (2.4)$$

Where $Re(s) < -1$

Result – 3:

$$\Phi(a_1 z, s, a) = \sum_{k=0}^\infty \frac{(a_1 z)^{\frac{k(k+1)}{2}}}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-\left\{a + \frac{k(k+1)}{2}\right\} t}}{1 - a_1 z e^{-t}} \left(1 - (a_1 z)^{(k+1)} e^{-(k+1)t}\right) dt \quad \ldots (2.5)$$

$Re(s) > 1, |z| < 1, Re(a) > 0,$

In particular

$$\Phi(a_1 z, -s, a) = \sum_{k=0}^\infty \frac{(a_1 z)^{\frac{k(k+1)}{2}}}{\Gamma(-s)} \int_0^\infty \frac{t^{-s-1} e^{-\left\{a + \frac{k(k+1)}{2}\right\} t}}{1 - a_1 z e^{-t}} \left(1 - (a_1 z)^{(k+1)} e^{-(k+1)t}\right) dt \quad \ldots (2.6)$$

Where $Re(s) < -1$

Result – 4:

$$\Phi(a_1 z, s, a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-at}}{1 - (a_1 z)^2 e^{-2t}} dt + \frac{(a_1 z)}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-(a+1)t}}{1 - (a_1 z)^2 e^{-2t}} dt \quad \ldots (2.7)$$

$Re(s) > 1, |z| < 1, Re(a) > 0,$

In particular

$$\Phi(a_1 z, -s, a) = \frac{1}{\Gamma(-s)} \int_0^\infty \frac{t^{-s-1} e^{-at}}{1 - (a_1 z)^2 e^{-2t}} dt + \frac{(a_1 z)}{\Gamma(-s)} \int_0^\infty \frac{t^{-s-1} e^{-(a+1)t}}{1 - (a_1 z)^2 e^{-2t}} dt \quad \ldots (2.8)$$

$Re(s) < -1, |z| < 1, Re(a) > 0,$

Result – 5:

$$\Phi(a_1 z, s, a) = \frac{1}{\Gamma(s)} \sum_{k=0}^\infty \frac{t^{s-1} e^{-(a+kp)t} (a_1 z)^{kp}}{1 - a_1 z e^{-t}} (1 - (a_1 z)^p e^{-pt}) dt \quad \ldots (2.9)$$

$Re(s) > 1, p \in z^+, |z| < 1, Re(a) > 0,$

In particular

$$\Phi(a_1 z, -s, a) = \frac{1}{\Gamma(-s)} \sum_{k=0}^\infty \frac{t^{-s-1} e^{-(a+kp)t} (a_1 z)^{kp}}{1 - a_1 z e^{-t}} (1 - (a_1 z)^p e^{-pt}) dt \quad \ldots (2.10)$$

Where $Re(s) < -1$

Prof of (2.1) and (2.2)

To prove the result in (2.1) we have the generalized Zeta function defined in (1.2)

$$\Phi(a_1 z, s, a) = \sum_{n=0}^\infty \frac{(a_1 z)^n}{(a+n)^s}$$

It can be written in the following form

$$\Phi(a_1 z, s, a) = \sum_{n=0}^r \frac{(a_1 z)^n}{(a+n)^s} + \sum_{n=r+1}^\infty \frac{(a_1 z)^n}{(a+n)^s} \quad \ldots (2.11)$$

On using the definition of Gamma function and changing the order of summation and integration, we have

$$\Phi(a_1 z, s, a) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-at} t^{s-1} \left\{\sum_{n=0}^r (a_1 z)^n e^{-nt}\right\} dt + \frac{1}{\Gamma(s)} \int_0^\infty e^{-at} t^{s-1} \left\{\sum_{n=r+1}^\infty (a_1 z)^n e^{-nt}\right\} dt \quad \ldots (2.12)$$

On summing the inner series in view of (1.11) and (1.12), we at once arrive at the desired result in (2.1). The result in (2.2) is obtained on replacing s by –s in (2.1)

Prof of (2.3) and (2.4)

From result of (2.1)

$$\sum_{n=0}^{r}\frac{(a_1z)^n}{(a+n)^s}=\frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-at}}{1-a_1ze^{-t}}\left(1-(a_1z)^{(r+1)}e^{-(r+1)t}\right)dt \quad \text{......... (2.13)}$$

In the same way as the proof of (2.1), we have

$$\sum_{n=r+1}^{r+k}\frac{(a_1z)^n}{(a+n)^s}=\sum_{n=r+1}^{r+k}\frac{(a_1z)^n}{\Gamma(s)}\int_0^\infty e^{-at}t^{s-1}e^{-nt}\ dt \quad \text{......... (2.14)}$$

Changing the order of summation and integration than using the result in (1.11) we have

$$\sum_{n=r+1}^{r+k}\frac{(a_1z)^n}{(a+n)^s}=\frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-at}(a_1z)^{(r+1)}e^{-(r+1)t}}{1-a_1ze^{-t}}(1-(a_1z)^ke^{-kt})dt \quad \text{......... (2.15)}$$

And

$$\sum_{n=r+k+1}^{\infty}\frac{(a_1z)^n}{(a+n)^s}=\sum_{n=r+1}^{r+k}\frac{(a_1z)^n}{\Gamma(s)}\int_0^\infty e^{-at}t^{s-1}e^{-nt}\ dt$$

Changing the order of summation and integration, we have

$$\sum_{n=r+k+1}^{\infty}\frac{(a_1z)^n}{(a+n)^s}=\frac{1}{\Gamma(s)}\int_0^\infty e^{-at}t^{s-1}\left\{\sum_{n=r+k+1}^{\infty}(a_1z)^ne^{-nt}\right\}dt$$

on using the result in (1.12)

$$\sum_{n=r+k+1}^{\infty}\frac{(a_1z)^n}{(a+n)^s}=\frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-at}(a_1z)^{(r+k+1)}e^{-(r+k+1)t}}{1-a_1ze^{-t}}dt \quad \text{......... (2.16)}$$

We have (2.3) after summing (2.13), (2.15) and (2.16) since

$$\Phi(a_1z,s,a)=\left(\sum_{n=0}^{r}+\sum_{n=r+1}^{r+k}+\sum_{n=r+k+1}^{\infty}.\right)\frac{(a_1z)^n}{(a+n)^s}$$

$(r\in z_0^+), k\in z^+$

The result in (2.4) is obtained on replacing s by –s in (2.3)
Prof of (2.5) and (2.6)

In the right part of (1.1) put n=0 and using of definition of Gamma function we get

$$\frac{1}{a^s}=\frac{1}{\Gamma(s)}\int_0^\infty e^{-at}t^{s-1}\ dt \quad \text{......... (2.17)}$$

And

$$\sum_{n=1}^{2}\frac{(a_1z)^n}{(a+n)^s}=\frac{a_1z}{(a+1)^s}+\frac{(a_1z)^2}{(a+2)^s}$$

$$\sum_{n=1}^{2}\frac{(a_1z)^n}{(a+n)^s}=\frac{a_1z}{\Gamma(s)}\int_0^\infty e^{-(a+1)t}t^{s-1}\ dt+\frac{(a_1z)^2}{\Gamma(s)}\int_0^\infty e^{-(a+2)t}t^{s-1}\ dt$$

$$\sum_{n=1}^{2}\frac{(a_1z)^n}{(a+n)^s}=\frac{a_1z}{\Gamma(s)}\int_0^\infty e^{-(a+1)t}t^{s-1}\left\{\frac{1+a_1ze^{-t}}{1-a_1ze^{-t}}\right\}(1-a_1ze^{-t})dt$$

$$\sum_{n=1}^{2}\frac{(a_1z)^n}{(a+n)^s}=\frac{a_1z}{\Gamma(s)}\int_0^\infty e^{-(a+1)t}t^{s-1}\left\{\frac{1-(a_1z)^2e^{-2t}}{1-a_1ze^{-t}}\right\}\ dt \quad \text{......... (2.18)}$$

Similarly

$$\sum_{n=3}^{5}\frac{(a_1z)^n}{(a+n)^s}=\frac{(a_1z)^3}{\Gamma(s)}\int_0^\infty e^{-(a+3)t}t^{s-1}\left\{\frac{1-(a_1z)^3e^{-3t}}{1-a_1ze^{-t}}\right\}dt \quad \text{....... (2.19)}$$

$$\sum_{n=6}^{9}\frac{(a_1z)^n}{(a+n)^s}=\frac{(a_1z)^4}{\Gamma(s)}\int_0^\infty e^{-(a+6)t}t^{s-1}\left\{\frac{1-(a_1z)^4e^{-4t}}{1-a_1ze^{-t}}\right\}dt \quad \text{....... (2.20)}$$

And so on for $a > 0$ and $Re(s) > 1$, then after the summing up (2.19) and (2.20) on wards. We get (2.5)
Since

$$\Phi(a_1z,s,a)=\left(\sum_{n=0}^{0}+\sum_{n=1}^{2}+\sum_{n=3}^{5}+\sum_{n=6}^{9}+\ ....\right)\frac{(a_1z)^n}{(a+n)^s}$$

The result in (2.6) is obtained on replacing s by –s in (2.5)
Prof of (2.7) and (2.8)

To prove the result in (2.7), we can be written (1.2) in the following form

$$\Phi(a_1z,s,a)=\sum_{m=0}^{\infty}\frac{(a_1z)^{2m}}{(a+2m)^s}+\sum_{m=0}^{\infty}\frac{(a_1z)^{2m+1}}{(a+2m+1)^s} \quad \text{....... (2.21)}$$

On using (1.10), we have

$$\Phi(a_1z,s,a)=\sum_{m=0}^{\infty}(a_1z)^{2m}\frac{1}{\Gamma(s)}\int_0^\infty e^{-(a+2m)t}t^{s-1}\,dt+\sum_{m=0}^{\infty}(a_1z)^{2m+1}\frac{1}{\Gamma(s)}\int_0^\infty e^{-(a+2m+1)t}t^{s-1}\,dt \quad \text{....... (2.22)}$$

Changing the order of summation and integration

$$\Phi(a_1z,s,a)=\frac{1}{\Gamma(s)}\int_0^\infty e^{-at}t^{s-1}\{\textstyle\sum_{m=0}^{\infty}(a_1ze^{-t})^{2m}\}dt+\frac{1}{\Gamma(s)}\int_0^\infty e^{-at}t^{s-1}\{\textstyle\sum_{m=0}^{\infty}(a_1ze^{-t})^{2m+1}\}\,dt \quad \text{....... (2.23)}$$

On summing the inner series in view of (1.12), we at once arrive the desired result in (2.7)
The result in (2.8) is obtained on replacing s by –s in (2.7)

Prof of (2.9) and (2.10)
To prove the result in (2.9), we can write

$$\sum_{n=0}^{p-1}\frac{(a_1z)^n}{(a+n)^s}=\sum_{n=0}^{p-1}\frac{(a_1z)^n}{\Gamma(s)}\int_0^\infty e^{-(a+n)t}t^{s-1}\,dt \quad \text{....... (2.24)}$$

Changing the order of summation and integration and after using (1.11), we have

$$\sum_{n=0}^{p-1}\frac{(a_1z)^n}{(a+n)^s}=\frac{1}{\Gamma(s)}\int_0^\infty e^{-at}t^{s-1}\left\{\frac{1-(a_1ze^{-t})^p}{1-a_1ze^{-t}}\right\}dt \quad \text{....... (2.25)}$$

Similarly

$$\sum_{n=p}^{2p-1}\frac{(a_1z)^n}{(a+n)^s}=\sum_{n=p}^{2p-1}\frac{(a_1z)^n}{\Gamma(s)}\int_0^\infty e^{-(a+n)t}t^{s-1}\,dt$$

$$\sum_{n=p}^{2p-1}\frac{(a_1z)^n}{(a+n)^s}=\frac{1}{\Gamma(s)}\int_0^\infty e^{-at}t^{s-1}\{\textstyle\sum_{n=p}^{2p-1}(a_1ze^{-t})^n\}\,dt \quad \text{....... (2.26)}$$

On using (1.11)

$$\sum_{n=p}^{2p-1}\frac{(a_1z)^n}{(a+n)^s}=\frac{1}{\Gamma(s)}\int_0^\infty e^{-(a+p)t}t^{s-1}\left\{\frac{(a_1z)^p[(1-(a_1ze^{-t})^p]}{1-a_1ze^{-t}}\right\}dt \quad \text{....... (2.27)}$$

And

$$\sum_{n=2p}^{3p-1}\frac{(a_1z)^n}{(a+n)^s}=\frac{1}{\Gamma(s)}\int_0^\infty e^{-(a+2p)t}t^{s-1}\left\{\frac{(a_1z)^{2p}[(1-(a_1ze^{-t})^p]}{1-a_1ze^{-t}}\right\}dt \quad \text{....... (2.28)}$$

By induction, we know that the (k+1)th summation is

$$\sum_{n=kp}^{(k+1)p-1} \frac{(a_1 z)^n}{(a+n)^s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-(a+kp)t} t^{s-1} \left\{ \frac{(a_1 z)^{kp}\left[\left(1-\left(a_1 z e^{-t}\right)^p\right]}{1-a_1 z e^{-t}} \right\} dt \quad \text{(2.29)}$$

For $Re(a) > 0, k \in Z_0^+, p \in Z^+$

After the summation up to (2.25)~(2.29), we get (2.9) under the conditions.
The result in (2.10) is obtained on replacing s by –s in (2.9)

Special Cases

(1)If in (2.1) to (2.10), we take z=1 and a_1=1, then these results provided the known integral representations of generalized Hurwitz Zeta function $\zeta(s, a)$ respectively [4, pp. 9-95, eqs. (3), (4); (12), (13); (20), (21); (28), (29); (32), (33)].

REFERENCES

Erdélyi, A, Magnus, W., Oberhettinger, F. and Tricomi, F.G. 1953. *Higher Transcendental Function,* Vol 1, Mc Graw Hill, New York, Toronto and London.

Goyal, S. P. and Laddha, R. K. 1997. *On the generalized Reimann Zeta function and the generalized Lambert transform*, *GanitaSandesh*, Vol II, No. 2, pp. 99-108.

Gradshteyn, I.S. and Ryzhik, I. M. 2002. *Table of Integrals Series and Products. 6th edition*, Academic press London.

Nishimoto, K., Yen, C. E. and Lin, M. L. 2002. Some integral forms for a generalized Zeta function, *J. Frac. Cal.* ,Vol 22, pp. 91-97.

Srivastava H. M. and Choi, J. 2001. *Series Associated with the Zeta and Related Functions*, Kluwer Academic Publishers, Dordrecht, Boston and London.

Srivastava H. M. and Choi, J. 2012. *Zeta and q-Zeta Functions and Associated Series and Integrals*.

****