Available online at http://www.journalcra.com

International Journal of Current Research
Vol. 7, Issue, 12, pp.24090-24097, December, 2015

INTERNATIONAL JOURNAL OF CURRENT RESEARCH

ISSN: 0975-833X

RESEARCH ARTICLE

A WAY FOR CONSTRUCTING HYBRID METHODS WITH HIGH ORDER OF ACCURACY AND THEIR APPLICATION TO SOLVING OF ODE OF FIRST ORDER

[*,1]Mehdiyeva, G., [1,2]Ibrahimov, V. and [1]Imanova, M.

[1]Baku State University, Department of Computational Mathematics, Baku, Azerbaijan
[2]Institute of Control Systems named after Academician A. Huseynov, Baku, Azerbaijan

ARTICLE INFO

Article History:
Received 18th September, 2015
Received in revised form
25th October, 2015
Accepted 17th November, 2015
Published online 30th December, 2015

Key words:
Initial value problem,
ODE,
Hybrid methods,
Stable, convergence,
Predictor-corrector schemes.

ABSTRACT

It is known, that many phenomena of neutrality are reduced to solving ordinary differential equation (ODE). There are several papers dedicated to solving ODE. In this paper, which compares many known algorithms applied to solving differential equations, also suggested an algorithm that uses hybrid methods and give a procedure for constructing higher order of accuracy hybrid methods. Concrete methods are constructed, and their advantages are indicated. In particularly suggested stable methods with the order of accuracy $p \le 6$ using two mesh points (for $k = 1$)

***Citation*: Mehdiyeva, G., Ibrahimov, V. and Imanova, M., 2015.** "A way for constructing hybrid methods with high order of accuracy and their application to solving of ode of first order", *International Journal of Current Research*, 7, (12), 24090-24097.

INTRODUCTION

There is a wide arsenal of numerical methods for solving ordinary differential equation of first order. One of these methods, connected to Clairaut, used the indirect-numerical method to investigate the orbit of the comet Galileo in Subbotin (1937). Like Clairaut, many scientists have since applied the indirect-numerical method for investigating practical problems. However, Euler determined the shortcomings of existing methods and constructed a direct method; this method is now appropriately called the Euler method. Euler also determined the shortcomings of his method and suggested two ways to correct the indicated deficiency in Euler (1956). One of them is the use of Taylor's formula. Substituting the higher-order derivatives in Taylor's formula with the first derivative, Runge-Kutta and Adams constructed numerical methods that generated the one and multistep methods. Consider the following initial-value problem:

$$y' = f(x, y), \; y(x_0) = y_0. \qquad \text{..............................} (1)$$

The aim of our paper is to construct a numerical method for calculating approximate solutions to problem (1).

Corresponding author: Mehdiyeva,
Baku State University, Department of Computational Mathematics, Baku, Azerbaijan.

Therefore, we suppose that problem (1) has a unique solution determined on the segment $[x_0, X]$ by the constant step-size $0 < h$. To construct numerical method, we partition the segment $[x_0, X]$ divided into N equal parts and determine the mesh points given by the following form: $x_i = x_0 + ih$, $i = 0,1,...,N$. Denote by y_i the approximated values solution and by the $y(x_i)$ exact values of the solution of problem (1) at the point x_i. The approximate values of the function $f(x, y)$ at the point (x_m, y_m) will be given by the following description:

$f_m = f(x_m, y_m), \;\; m = 0,1,2,..., N$ It is known that the Runge-Kutta method, when applied to the solution of problem (1), may be written in its general form as follows:

$$y_{n+1} = y_n + h\sum_{i=1}^{s} b_i K_i^{(s)}, \qquad \text{....................} (2)$$

$$K_i^{(s)} = f(x_n + \alpha_i h, y_n + h(\beta_{1,s} K_1^{(s)} + \beta_{2,s} K_2^{(s)} + \ldots + \beta_{s,s} K_s^{(s)})), \quad i = 1,2,\ldots,s$$

Usually this method is called the Runge-Kutta implicit method. The generalization of the Adams method may be written in the following form:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i}\,. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

In references, this method is called the k-step method with constant coefficients. As has been noted, Euler's method follows from the formula (3) as a particular case. However, methods (2) and (3) have different properties; each has its advantages and shortcomings. In the middle of the XX century, scientists constructed procedures called hybrid methods that are capable of preserving the best properties of both the one and multistep methods in Skvortsov (2009) and Mehdiyeva *et al*. (2005). Recently, these methods have received priority. Before we provide a scheme for constructing hybrid methods with improved properties, we would like to give some brief information on the development of approximately methods in Mehdiyeva *et al*. (2011) and Butcher (2008). One of the papers devoted to the construction of hybrid methods is attributable to Gear (1965), Butcher (1965) and Gupta (1979). In its general form, this method may be written as follows:

$$\sum_{i=0}^{k} a_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i} + h\beta f_{n+k-\nu} \quad (0 < \nu < 1). \qquad \ldots\ldots\ldots\ldots\ldots (4)$$

Gear's method is a hybrid method of type (3). One of the first hybrid method of type (1) was constructed in Dahlquist (1956) and has the following form:

$$y_{n+1} = y_n + h(K_1 + K_2)/2\,, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Where

$$K_1 = f(x_n + \frac{3-\sqrt{3}}{6}h,\ y_n + h(K_1 + \frac{3-2\sqrt{3}}{3}K_2)/4);$$

$$K_2 = f(x_n + \frac{3+\sqrt{3}}{6}h,\ y_n + h(\frac{3+2\sqrt{3}}{3}K_1 + K_2)/4).$$

Often, a hybrid method has a symmetric form. Similar methods may be written in the following general form:

$$\sum_{i=0}^{k} a_i y_{n+i} = h \sum_{i=0}^{k-1} \beta_i f_{n+i} + h\beta_0 f_{n+s} + h\beta_k f_{n+k-s}, \quad (0 < s < 1). \qquad \ldots\ldots (6)$$

This method is more precise than the corresponding classic Runge-Kutta and Adams methods. Note that the method used here is a generalization of the methods cited above. If we generalize the aforementioned methods, we obtain the following:

$$\sum_{i=0}^{k} a_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i+l_i} \quad (|l_i| < 1,\ i = 0,1,2,\ldots,k). \qquad \ldots\ldots (7)$$

It is easy to see that for $l_i = 0,\ (i = 0,1,2,\ldots,k)$ the established k-step method of type (3) with constant coefficients follows from method (7). However, if $l_i \neq 0,\ (0 \le i \le k)$, then from (7) we will obtain hybrid methods. For example, take the following symmetric hybrid method:

$$y_{n+1} = y_n + h(f_{n+1/2-\sqrt{3}/6} + f_{n+1/2+\sqrt{3}/6})/2. \qquad \ldots\ldots\ldots\ldots\ldots (8)$$

Developing this idea, here we also investigate the generalized form of hybrid methods based on formula (7). More precisely, we consider the following:

$$\sum_{i=0}^{k} a_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i} + h \sum_{i=0}^{k} \gamma_i f_{n+i+l_i} \quad (|l_i| < 1). \qquad \ldots\ldots\ldots (9)$$

Obviously, method (9) fits multistep methods with constant coefficients and multistep hybrid methods. Note that the development of hybrid methods has been stated in chronological order. Therefore, in section 2, we consider strategies deriving from (2), whereas in section 3 we investigate methods based on the formula in (9) are investigated. In these two sections, exact methods with degree $4 \le p \le 9$ are provided. Section 4 is devoted to the construction of algorithms applying the methods constructed here. Note that the integer-valued quantity p is called the degree of method (7) if the following asymptotic equality holds:

$$\sum_{i=0}^{k} (a_i y(x + ih) - h\beta_i y'(x + (i + l_i)h)) = O(h^{p+1}),\ h \to 0. \qquad \ldots\ldots\ldots\ldots$$

(10)

On a method for constructing hybrid methods with the higher order of accuracy

As noted above, hybrid methods possess some improved features over the Runge-Kutta and Adams methods. However, their accuracy and the boundaries of the stability domain depend on the variables $\alpha_i,\ \beta_i,\ l_i\ (i = 0,1,2,\ldots,k)$. Therefore, we consider the definition of these variables, and to this end we will employ the method of undetermined coefficients. Usually, the use of this method is based on Taylor's formula, which in this case takes the following form:

$$y(x + ih) = y(x) + ihy'(x) + \frac{(ih)^2}{2!}y''(x) + \ldots + \frac{(ih)^p}{p!}y^{(p)}(x) + O(h^{p+1}), \qquad \ldots\ldots\ldots (11)$$

$$y'(x+\gamma_i h) = y'(x) + \gamma_i h y''(x) + \frac{(\gamma_i h)^2}{2!} y'''(x) + ... + \frac{(\gamma_i h)^{p-1}}{(p-1)!} y^{(p-1)}(x) + O(h^p) \quad \text{............ (12)}$$

Where $x = x_0 + nh$ is a fixed point.

To calculate the values of the parameters $\alpha_i, \beta_i, l_i \ (i = 0,1,2,...,k)$, we account for equations (11) and (12) in the asymptotic equation (10). Then, we will have:

$$\sum_{i=0}^{k} \alpha_i = 0\,; \quad \sum_{i=0}^{k} \frac{i^\nu}{\nu!} \alpha_i = \sum_{i=0}^{k} \frac{\gamma_i^{\nu-1}}{(\nu-1)!} \beta_i \quad (\nu = 1,2,...,p). \quad (13)$$

Here, $\gamma_i = i + l_i \quad (i = 0,1,2,..., k)$.

Thus, to calculate the parameters $\alpha_i, \beta_i, l_i \quad (i = 0,1,2,...,k)$, we have obtained a system of $p+1$ algebraic equations in $3k+3$ unknowns. Obviously, the system of equations in (13) will always have the trivial solution. However, the trivial solution of (13) is not of interest. Therefore, consider the case when system (13) has a nontrivial solution. It is known that for the case:

$$p < 3k + 2 \quad \text{.................................... (14)}$$

the system (13) has a non-zero solution.

However, if we consider the case for which $l_i = 0 \ (i = 0,1,2,...,k)$, then the stable k-step method with constant coefficients, whose degree is given by the following:

$$p_{\max} = 2k\,. \quad \text{...................................... (15)}$$

arises from (3).

The straightforward comparison of (14) and (15) demonstrates that hybrid methods of type (7) are more precise and give rise to investigative method (7). It is known that the basic properties of numerical methods are defined by the values of their coefficients; we impose some restrictions on the coefficients of method (7). These conditions are the analogues of appropriate conditions imposed on the coefficients of the multistep method in (3) (see (12)):

A: The coefficients $\alpha_i, \beta_i \ (i = 0,1,2,...,k)$ are all real numbers; moreover, $\alpha_k \neq 0$.

B: The characteristic polynomials

$$\rho(\lambda) \equiv \sum_{i=0}^{k} \alpha_i \lambda^i\,, \quad \nu(\lambda) \equiv \sum_{i=0}^{k} \beta_i \lambda^{i+l_i};$$

have no common multiple different from constant.

C: $\sigma(1) \neq 0$ and $p \geq 1$.

Condition A arises from the fact that the solution of the problem under consideration requires a real-valued function, and requiring that $\alpha_k \neq 0$ provides the values of $y_{n+k} \ (n = 0,1,2,...)$. Next, suppose that condition B does not hold. Then, it follows that the polynomials $\rho(\lambda)$ and $\upsilon(\lambda)$ have a nontrivial common multiple different from constant, which we denote by $\varphi(\lambda)$. Accounting for this, and using the shift operator $E(E^\nu y(x) = y(x + \nu h))$, we may rewrite difference equation (7) in the following form:

$$\rho(E) y_n - h\upsilon(E) y'_n = 0. \quad \text{........................... (16)}$$

Under our assumptions, we may rewrite equation (7) as follows:

$$\varphi(E)(\rho_1(E) y_n - h\upsilon_1(E) y'_n) = 0. \quad \text{.............. (17)}$$

Here,

$$\rho_1(\lambda) = \rho(\lambda) / \varphi(\lambda); \ \ \upsilon_1(\lambda) = \upsilon(\lambda) / \varphi(\lambda).$$

Hence, we obtain the following because $\varphi(\lambda) \neq const$:

$$\rho_1(E) y_n - h\upsilon_1(E) y'_n = 0. \quad \text{............................. (18)}$$

It is obvious that for the difference equation given in (18) to have a unique solution, at most $k-1$ initial data points should be provided. However, it is known from the theory of difference equations that for a difference equation of order k to have a unique solution, k initial data points should be provided. However, the difference equations in (18) and (7) are equivalent. Hence, we deduce that difference equation (7) has a unique solution when only $k-1$ initial data are known, and this contradicts the aforementioned theory. Consequently, our assumption that there exists a common multiple of the polynomials $\rho(\lambda)$ and $\vartheta(\lambda)$ cannot be true. Thus, condition B must hold for the application of method (7). Now, consider the validity of condition C. Suppose that method (7) is converges. Then, passing to equation (7) as $h \to 0$, we have:

$$\sum_{i=0}^{k} \alpha_i y(x) = 0, \quad \text{......................... (19)}$$

Here $x = x_0 + nh$ is a fixed point. If we take into account our assumption that $y(x) \neq 0$, then from (19) we have

$$\rho(1) = 0, \quad \text{............................... (20)}$$

and this provides us with a necessary convergence condition. Allowing for this condition, we can write:

$$\rho(\lambda) = (\lambda - 1)\rho_1(\lambda)$$

Taking into account the above obtained in relation (16), we have:

$$\rho_1(E)(y_{j+1}-y_j)-h\upsilon(E)y'_y=0. \qquad \text{……………. (21)}$$

We now range the values of j from 0 to n, sum the obtained equations and in results receive the following:

$$\rho_1(E)(y_{n+1}-y_0)-h\upsilon(E)\sum_{j=0}^{n}y'_j=0. \qquad \text{…………... (22)}$$

Taking the limit of this equation as $h\to 0$ yields:

$$\rho_1(1)(y(x)-y_0)=\upsilon(1)\int_{x_0}^{x}y'(\xi)d\xi. \qquad \text{……………... (23)}$$

However, from (1) we know that

$$y(x)=y(x_0)+\int_{x_0}^{x}f(\xi,y(\xi))d\xi. \qquad \text{………………… (24)}$$

Comparing equations (23) and (24) gives us the following:

$$\rho_1(1)=\rho'(1)=\upsilon(1).$$

It is easy to show that from the following conditions:

$$\rho(1)=0;\rho'(1)=\upsilon(1), \qquad \text{………………… (25)}$$

receive the condition $p\geq 1$.

Now, we will prove that $\upsilon(1)\neq 0$. So suppose the opposite is true. Then, from our conditions that $\rho(1)=0$ and $\rho'(1)=0$, we obtain that $\lambda=1$ is a two multiple root of polynomial $\rho(\lambda)$. Consider the following homogeneous difference equation:

$$\alpha_k y_{n+k}+\alpha_{k-1}y_{n+k-1}+...+\alpha_1 y_{n+1}+\alpha_0 y_n=0, \qquad \text{……. (26)}$$

Its general solution may be written in the following form:

$$y_m=c_1\lambda_1^m+c_1 m\lambda_1^m+c_3\lambda_3^m+...+c_k\lambda_k^m, \ (27)$$

Here, $\lambda_i\ (i=1,2,...,k)$ represent the roots of polynomial $\rho(\lambda)$. As $h\to 0$, it follows that $y_m\to\infty$, because $m\to\infty$. Thus, we obtain that if $\upsilon(1)=0$, then the method will not converge. Hence, it follows that we should have $\upsilon(1)\neq 0$. Therefore, in what follows we will assume that the coefficients $\alpha_i,\beta_i\ (i=0,1,2,...,k)$ from method (7) satisfy all three conditions A, B, C. Consider to the construction methods of type (7) and suppose that $k=1$. Then, under the assumption that $\alpha_1=-\alpha_0=1$, we will have the following system of equations for the variables β_0,β_1,l_0 and l_1:

$$\begin{aligned}&\beta_0+\beta_1=1,\\ &l\beta_0+\gamma\beta_1=1/2,\\ &l^2\beta_0+\gamma^2\beta_1=1/3,\\ &l^3\beta_0+\gamma^3\beta_1=1/4.\end{aligned} \qquad \text{…………………………… (28)}$$

Here, $l=l_0$, and $\gamma=1+l_1$. Solving this nonlinear system of equations for l results in the following quadratic equation:

$$l^2-l+1/6=0.$$

The value of γ is determined from the equation $\gamma+l=1$. Note that the method with the degree $p=4$ can be written as follows:

$$y_{n+1}=y_n+h(f_{n+l_0}+f_{n+1+l_1})/2. \qquad \text{…………………(29)}$$

Here, $l_1=-l_0$; $l_0=(3-\sqrt{3})/6$, $1+l_1=(3+\sqrt{3})/6$.

To apply method (29), we should also know the value of the y_{n+l_0} and $y_{n+\gamma}$. Note that these variables are independent from y_{n+1}, because that method (29) is explicit. But therefore, there still exist implicit hybrid methods. For example, consider the following method in Makroglou (1982):

$$y_{n+1}=y_n+h(3f_{n+1/3}+f_{n+1})/4. \qquad \text{……………… (30)}$$

This method is an implicit hybrid method with degree $p=3$ and is A-stable (see (13)). Note that the coefficients of method (30) satisfy the system in (28) except for its final equation. When constructing an algorithm for using method (29), it can be shown that from the standpoint of application, method (29) has some advantages. Now consider the case $\beta_k\neq 0$ and $l_k\neq 0$. If $l_k>0$, then from method (7) we will obtain a forward-jumping hybrid method. It can be observed from method (29) and (30) that in hybrid methods, the solutions of problem (1) may participate as the mesh points $x_i(i=0,1,2,...)$ and as the intermediate points $x_{i+l_j}\ (i=0,1,2,...,j=0,1,2,...k)$. Therefore, let us consider some generalizations of method (7).

On a scheme for construction of hybrid methods

As noted above, hybrid methods may be generalized by the form in (9). It is easy to see that using the established k-step algorithm with constant coefficients and assuming that $l_i=0,\ (i=0,1,2,...k)$, and $\beta_i=0,\ (i=0,1,2,...k)$, method (3) may be obtained from method (9). The stability and degree of method (9) may be calculated according to the above definitions. One of the basic issues with evaluating this algorithm is that of defining the relationship between its degree and order. Before establishing this relationship, let us consider some restrictions imposed on coefficients of the method (19).

A: The coefficients $\alpha_i,\beta_i,\gamma_i,l_i,\ (i=0,1,2,...,k)$ must be real numbers; moreover, $\alpha_k\neq 0$.

B: The characteristic polynomials

$$\rho(\lambda) \equiv \sum_{i=0}^{k} \alpha_i \lambda^i ,\nu(\lambda) \equiv \sum_{i=0}^{k} \beta_i \lambda^i ; \gamma(\lambda) \equiv \sum_{i=0}^{k} \gamma_i \lambda^{i+l_i}$$

have no common multiple different from constant.

C: $\sigma(1) + \gamma(1) \neq 0$ and $p \geq 1$.

The necessity of conditions A and B is proved similarly to the case that $\beta_i = 0,\ (i = 0,1,2,...,k)$. Therefore, we consider condition C and assume that method (9) is converges. Then, following the same steps from section 2, we find that

$$\rho(1) = 0. \qquad \text{.......} (31)$$

Substituting the condition obtained in the derivation of method (9), we have that

$$\rho_1(E)(y_{j+1} - y_j) - h\upsilon(E) f_j - h\gamma(E) f_j = 0. \qquad \text{......} (32)$$

Here, as in section 1, after summing the resulting equations over all j ranging from 0 to n, we will find that

$$\rho_1(E)(y_{n+1} - y_0) = (\upsilon(E) + \gamma(E))h\sum_{j=0}^{n} f_j. \qquad \text{.........} (33)$$

We now take the limit of this equation as $h \to 0$ and obtain that

$$\rho_1(1)(y(x) - y_0) = (\upsilon(1) + \gamma(1))\int_{x_0}^{x} f(\xi, y(\xi))d\xi. \qquad \text{......} (34)$$

Hence, it follows that

$$\rho_1(1) = \vartheta(1) + \gamma(1). \qquad \text{.......} (35)$$

In turn, we obtain the following:

$$\rho(1) = 0,\ \rho'(1) = \vartheta(1) + \gamma(1).$$

Consequently, $p \geq 1$. The necessity of the condition $\sigma(1) + \gamma(1) \neq 0$ rests on convergence and follows a proof similar to the one derived in section 1. Now, by the method of undetermined coefficients, we must examine the definition of the quantities $\alpha_i, \beta_i, \gamma_i, l_i\ (i = 0,1,2,...,k)$, so we will consider the following expansion:

$$y'(x + lh) = y'(x) + lhy''(x) + \\ + \frac{(lh)^2}{2!} y'''(x) + ... + \frac{(lh)^{p-1}}{(p-1)!} y^{(p)}(x) + O(h^p), \qquad \text{..........} (36)$$

Note that the values of the indicated quantities in some sense is connected with the relationship between the order and degree of method (9); therefore, we require the following lemma.

Lemma. Let $y(x)$ be a sufficiently smooth function, and assume that conditions A, B, and C are holds. For method (9) to have degree p, satisfies the following contains are necessary and sufficient:

$$\sum_{i=0}^{k} \alpha_i = 0,\ \sum_{i=0}^{k} i\alpha_i = \sum_{i=0}^{k} (\beta_i + \gamma_i), \\ \sum_{i=0}^{k} \frac{i^{l-1}}{(l-1)!}\beta_i + \sum_{i=0}^{k} \frac{(i+l_i)^{l-1}}{(l-1)!}\gamma_i = \sum_{i=0}^{k} \frac{i^l}{l!}\alpha_i \qquad \text{................} (37) \\ (l = 2,3,...,p).$$

Proof. We first prove that if method (9) has degree p, then the coefficients $\alpha_i, \beta_i, \gamma_i, l_i \quad (i = 0,1,2,...,k)$ will satisfy the system of nonlinear algebraic equations given in (37). Using the degree of correlation (10), we can write the following:

$$\sum_{i=0}^{k} \alpha_i y(x + ih) - h\sum_{i=0}^{k} (\beta_i y'(x + ih) + \\ + \gamma_i y'(x + h(i + l_i))) = O(h^{p+1}), h \to 0 \qquad \text{..........} (38)$$

The left side of asymptotic equation (38) integrates equations (11), (12) and (36). We then have that

$$\left(\sum_{i=0}^{k} \alpha_i\right) y(x) + h\sum_{i=0}^{k} \left(i\alpha_i - \beta_i - \gamma_i\right) y'(x) + \\ + h^2 \sum_{i=0}^{k} \left(\frac{i^2}{2}\alpha_i - i\beta_i - (i + l_i)\gamma_i\right) y''(x) + ... + \\ + h^p \sum_{i=0}^{k} \left(\frac{i^p}{p!}\alpha_i - \frac{i^{p-1}}{(p-1)!}\beta_i - \frac{(i + l_i)^{p-1}}{(p-1)!}\gamma_i\right) y^{(p)}(x) = \\ = O(h^{p+1}), \qquad h \to 0. \qquad \text{................} (39)$$

Because method (9) has degree p, we obtain the following:

$$\sum_{i=0}^{k} \alpha_i y(x) + h\sum_{i=0}^{k} (i\alpha_i - \beta_i - \gamma_i) y'(x) + \\ + h^2 \sum_{i=0}^{k} (\frac{i^2}{2} - i\beta_i - (i + l_i)\gamma_i) y''(x) + ... \qquad \text{..} (40) \\ + h^p \sum_{i=0}^{k} (\frac{i^p}{p!}\alpha_i - \frac{i^{p-1}}{(p-1)!}\beta_i - \frac{(i + l_i)^{p-1}}{(p-1)!}\gamma_i) y^{(p)}(x) = 0.$$

It is known that $1, x, x^2, ..., x^p$ forms a linearly independent system; therefore, equation (40) is equivalent to the following:

$$\sum_{i=0}^{k} \alpha_i = 0,\ \sum_{i=0}^{k} (i\alpha_i - \beta_i - \gamma_i) = 0,...., \\ \sum_{i=0}^{k} (\frac{i^p}{p!}\alpha_i - \frac{i^{p-1}}{(p-1)!}\beta_i - \frac{(i + l_i)^{p-1}}{(p-1)!}\gamma_i) = 0, \qquad \text{.........} (41)$$

This coincides with the system of equations in (37). We now will prove that if the coefficients from method (9) is solution of the nonlinear system (37), then its degree is equal to p. Indeed, if we integrate equation (37) into equation (44), we obtain the asymptotic in (38). It follows from this asymptotic equation that method (9) must have degree p. It is easy to determine that for the chosen values $l_i = 0\ (i = 0,1,...,k)$, system (37) is in fact linear and coincides with the known system used for defining the coefficients of the multistep method with constant coefficients. Subject to the conditions $|l_0| + |l_1| + ... + |l_k| \neq 0$, system (37) is nonlinear. This homogeneous system contains from $p+1$ equations and $4k+4$ unknowns. It must possess the zero solution, and for system (37) to have a non-zero solution, the condition $4k+4 > p+1$ must be satisfied. Hence, we obtain that $p \le 4k+2$. Note that if we take $\beta_i = 0\ (i = 0,1,2,...,k)$, then the inequality tying together the degree and order of method (9) or (7) will take the following form:

$$p \le 3k+1.$$

Additionally, it is known that if we consider the case that all $\gamma_i = 0\ (i = 0,1,2,...,k)$, then the degree and order of method (9) must satisfies the condition $p \le 2k$. The relationship between the degree and order of hybrid methods shows that these methods are more precise than existing multistep methods. As noted above, if method (9) is stable and has the degree p, then $p \le 3k+3$. So if $p = 6$, then setting $k = 1$ yields a stable method. When $k = 2$, one can construct stable methods having degree $p = 8$ or $p = 9$. Consider method (9) for $k = 1$. In this case, assuming that $\alpha_1 = -\alpha_0 = 1$, system (37) takes the following form:

$$\begin{aligned}
&\beta_0 + \beta_1 + \gamma_0 + \gamma_1 = 1,\\
&\beta_1 + l_0\gamma_0 + l_1\gamma_1 = 1/2,\\
&\beta_1 + l_0^2\gamma_0 + l_1^2\gamma_1 = 1/3,\\
&\beta_1 + l_0^3\gamma_0 + l_1^3\gamma_1 = 1/4,\\
&\beta_1 + l_0^4\gamma_0 + l_1^4\gamma_1 = 1/5,\\
&\beta_1 + l_0^5\gamma_0 + l_1^5\gamma_1 = 1/6.
\end{aligned} \qquad \text{.........................} (42)$$

The solution of this nonlinear system yields the following:

$$\beta_0 = \beta_1 = 1/12,\ \gamma_0 = \gamma_1 = 5/12,$$
$$l_0 = 1/2 - \sqrt{5}/10,\ l_1 = 1/2 + \sqrt{5}/10.$$

The same method with degree $p = 6$ takes the following form:

$$\begin{aligned}
&y_{n+1} = y_n + h(f_{n+1} + f_n)/12 +\\
&+ 5h(f_{n+1/2-\sqrt{5}/10} + f_{n+1/2+\sqrt{5}/10})/12
\end{aligned} \qquad \text{..........} (43)$$

To apply hybrid methods to solving of some problems, we should know the values of $y_{n+1/2-\sqrt{5/10}}$ and $y_{n+1/2+\sqrt{5/10}}$, and the accuracy of these values should have order at least $O(h^6)$. Note that hybrid method (43) is implicit and that when applying it to solving of our initial problem in (1), a predictor-corrector scheme containing only one explicit method is used. Therefore, we consider the construction of an explicit method that (in one variant) has the following form:

$$\begin{aligned}
&y_{n+1} = y_n + hf_n/9 + h((16+\sqrt{6})f_{n+(6-\sqrt{6})/10} +\\
&+ (16-\sqrt{6})f_{n+(6+\sqrt{6})/10})/36.
\end{aligned} \qquad \text{......} (44)$$

This method is explicit and has degree $p = 5$. To use method (44) we must define $y_{n+\frac{6-\sqrt{6}}{10}}$ and $y_{n+\frac{6+\sqrt{6}}{10}}$. The technique used to calculate these quantities determines the properties of the block method. Suppose that the approximated values of the solution of problem (1) for $x_n + (1-\sqrt{5}/5)/2$ and $x_n + (1+\sqrt{5}/5)/2$ have been identified by some method. Then, (43) may be considered an equation in the unknowns y_{n+1}, whose solution is usually obtained via iterative processes. In contrast, we suggest a predictor-corrector method recalling block methods. It is easy to show that one can first calculate the values of y_{n+1} according to method (44) and then correct these values by the method (43). We therefore construct an algorithm for applying method (44) to solving of problem (1).

Construction of an algorithm that uses some hybrid methods

In this section, we will construct hybrid procedures for using methods having the degree $p = 4, p = 5$ and $p = 6$. Because these methods are constructed for $k = 1$ (the characteristic polynomial is rooted solely at $\lambda = 1$), all of them are stable. Methods (29) and (44) are explicit, whereas method (43) is implicit (as mentioned above). However, the application of explicit hybrid methods requires some additional auxiliary formulas. To this end, we construct an algorithm for method (29).

Algorithm 1. Applies method (29) to solving of problem (1).

Step 1. Calculate y_{n+l} and y_{n+1-l} by with the following block method:

$$\begin{aligned}
&y_{n+l} = y_n + lhf_n,\\
&\hat{y}_{n+l} = y_n + lh(f_n + \bar{f}_{n+l})/2,\\
&(\bar{f}_m = f(x_m, \bar{y}_m),\ \ m = 0,1,2,...),\\
&y_{n+l} = y_n + lh(5f_n + 8\hat{f}_{n+l})/12 -\\
&- lhf(x_n + 2lh, y_n + 2lh\hat{f}_{n+l})/12,\\
&(\hat{f}_m = f(x_m, \hat{y}_m),\ \ m = 0,1,2,...).
\end{aligned}$$

Repeat these schemes for $l := 1 - l$.

Step 2. Calculate y_{n+1} according to method (29). Here, we compute the values of the quantities y_{n+1} and y_{n+1-l} to within $O(h^4)$, which suffices for this algorithm. Now, we construct an algorithm for applying method (44).

Algorithm 2. Applies method (44) to the numerical solution of problem (1), assuming that the values y_0 and $y_{1/2}$ have been determined with the required accuracy.

Step I. Set $\hat{y}_{n+1} = y_n + hy'_{n+1/2}$,

Step II.

Set $y_{n+1} = y_n + h(\hat{y}'_{n+1} + 4y'_{n+1/2} + y'_n)/6$,

Step III. Set

$y_{n+3/2} = y_{n+1/2} + h(7y'_{n+1} - 2y'_{n+1/2} + y'_n)/6$, Step IV. Compute

$$\begin{aligned} y_{n+\alpha} = y_n + \alpha h y'_n + \alpha^2 h((\alpha^2 - 12\alpha + 6)y'_{n+3/2} - \\ -(3\alpha^2 - 48\alpha + 27)y'_{n+1} + \\ +(3\alpha^2 - 60\alpha + 54)y'_{n+1/2} - \\ -(\alpha^2 - 24\alpha + 33)y'_n)/18 \end{aligned}$$

for $\alpha = \dfrac{6-\sqrt{6}}{10}$ and $\alpha = \dfrac{6+\sqrt{6}}{10}$.

Step V. Conclude that

$$\begin{aligned} y_{n+1} = y_n + hf_n/9 + h((16+\sqrt{6})f_{n+(6-\sqrt{6})/10} + \\ +(16-\sqrt{6})f_{n+(6+\sqrt{6})/10})/36. \end{aligned}$$

Algorithm 3. Calculates the $y_{1/2}$.

$$\bar{y}_{1/2} = y_0 + \frac{h}{2}f_0,$$

$$\hat{y}_{1/2} = y_0 + h(f_0 + f(h/2, \bar{y}_{1/2}))/4,$$

$$\bar{y}_{n+1/6} = y_n + hf_n/6,$$

$$\hat{y}_{n+1/6} = y_n + h(f_n + \bar{f}_{n+1/6})/12,$$

$$\hat{y}_{n+1/2} = y_n + h(3\hat{f}_{n+1/6} + \hat{f}_{n+1/2})/4.$$

Algorithm 4. Approximates the solution of the initial-value problem

$$y' = f(x, y), \quad x_0 \le x \le X, \quad y(x_0) = y_0$$

at (N+1) equally spaced numbers taken from the interval $[x_0, X]$:

INPUT endpoints x_0, X; integer N;

Initial values y_0, $y_{1/2}$.

OUTPUT approximation y_i of $y(x_i)$ at each of the (N+1) values of x.

Step 1. Set $h = (x - x_0)/N$;

Step 2. For each $i = 1,2,...,N$, perform Steps 3-6.

Step 3. Set $\hat{y}_{i+1} = y_i + hf_{i+1/2}$;

$$y_{i+1} = y_i + h(\hat{f}_{i+1} + 4f_{i+1/2} + f_i)/6;$$

$$y_{i+3/2} = y_{i+1/2} + h(7\hat{f}_{i+1} - 2f_{i+1/2} + f_i)/6.$$

Step 4. For

$\alpha = (6-\sqrt{6})/10, (6+\sqrt{6})/10$, calculate

$$\begin{aligned} y_{i+\alpha} = y_i + \alpha h y'_i + \alpha^2 h((\alpha^2 - 12\alpha + 6)f_{i+3/2} - \\ -(3\alpha^2 - 48\alpha + 27)f_{i+1} + \\ +(3\alpha^2 - 60\alpha + 54)f_{i+1/2} - \\ -(\alpha^2 - 24\alpha + 33)f_i)/18. \end{aligned}$$

Step 5. Calculat

$$\begin{aligned} y_{i+1} = y_i + hf_i/9 + h((16+\sqrt{6})f_{i+(6-\sqrt{6})/10} + \\ +(16-\sqrt{6})f_{i+(6+\sqrt{6})/10})/36. \end{aligned}$$

Step 6. OUTPUT (i, y_i).

Step 7. STOP.

For demonstrate algorithm 1 consider to application of its to the solving of the next problem: $y' = \cos x,\ y(0) = 0$ (Exact solution as $y(x) = \sin x$).

Table 1. Results tabulated

Step size	Variable x	Error of the algorithm 4
$h = 0.05$	0.10	$0.14E-09$
	0.40	$0.56E-09$
	0.70	$0.93E-09$
	1.00	$0.12E-08$

Conclusion

We have constructed a multistep hybrid method with constant coefficients and some concrete hybrid methods of degree $4 \le p \le 6$ for $k = 1$. It is known that for $k = 1$, the k-step method with constant coefficients has maximal degree $p_{\max} = 2$, which yields a trapezoidal method. However, the hybrid approach constructed here has maximal degree $p_{\max} = 6$, although the application of the trapezoid method is simpler than applying a hybrid procedure. Using the Euler explicit method in place of the predictor method, one can construct a predictor-corrector scheme for the practical application of the trapezoid method. Remark, that for constricting stable methods with the degree $p = 2k + 2$ one can be used multistep methods with the second derivatives (see, for example Kobza (1975), Areo *et al.* (2008), Dahlquist (1959), Ibrahimov (2002), Mehdiyeva *et al.* (2012), Mehdiyeva *et al.* (2013), Sekar *et al.* (2011).

In this paper, we have constructed a block method for the construction of exact algorithms. This method, having degree $p = 4$, was described in algorithm 1, which assumed that the function $f(x, y)$ is calculated at each step; if necessary, the number of calculations of this function may be decreased. Note that after some modifications, algorithm 2 may realize for using method (43). The auxiliary formula featuring in the algorithms has been constructed via a Taylor formula with the appropriate accuracy. Therefore, these formulas may be simplified or replaced by more precise formulas. Method (9) offers a new approach for solving problem (1). Naturally, the establishment of relationships between the order and degree of a stable method (9), as well as the determination of the stability area and other relevant questions, hold scientific interest. We offer that hybrid methods are more promising (it is enough to remind the methods for $k=1$); therefore, there will be some necessary corrections and revisions during the course of their study. In the end, note that for $k=2$, we have constructed stable methods of type (9) with degree $p=8$ and $p=9$. Note that can be acquainted with application of hybrid methods to numerical solution of Volterra integral equation in Mehdiyeva *et al*. (2011) , with their application to the solution of integro-differential equation in Mehdiyeva *et al*. (2013). And in Kobza (1975) constructed hybrid method with degree $p=7$ for $k=3$ by using collocation approach.

Acknowledgment

The authors wish to express their thanks to academician Ali Abbasov for his suggestion to investigate the computational aspects of our problem and for his frequent valuable suggestion.

REFERENSES

Areo, E.A., R.A. Ademiluyi, Babatola P.O. 2008. Accurate collocation multistep method for integration of first order ordinary differential equations. *J. of Modern Math. and Statistics*, 2(1): 1-6, P. 1-6.

Butcher, J.C. 1965. A modified multistep method for the numerical integration of ordinary differential equations. *J. Assoc. Comput. Math*., v.12, pp.124-135.

Butcher, J.C. 2008. Numerical methods for ordinary differential equations. John Wiley and sons, Ltd, Second Edition, P. 463.

Dahlquist, G. 1956. Convergence and stability in the numerical integration of ordinary differential equations. *Math. Scand*., 4, p.33-53.

Euler, L. 1956. Integral calculus T.1, Gos.izd of engineering and technical literature, P.415.

Dahlquist, G. 1959. Stability and Error Bounds in the Numerical Integration of Ordinary Differential Equations. Trans. Of the Royal Inst. Of Techn. Stockholm, Sweden, Nr. 130, 87pp.

Gear, C.S. 1965. Hybrid methods for initial value problems in ordinary differential equations. SIAM, *J. Numer. Anal. v*. 2, pp. 69-86.

Gupta G.K. 1979. A polynomial representation of hybrid methods for solving ordinary differential equations. Mathematics of comp., volume 33, number 148, P.1251-1256.

Hammer, P.C., Hollingsworth, J. W. 1955. Trapezoildal methods of approximating solution of differential equations. MTAC-vol. 9, p.92-96.

Ibrahimov, V. 2002. On the maximal degree of the k-step Obrechkoff's method. *Bulletin of Iranian Mathematical Sociaty*, Vol.28, №1, p. 1-28.

Ibrahimov, V.R. 1982. On a nonlinear method for numerical calculation of the Cauchy problem for ordinary differential equation. Diff. equation and applications. Pron. of II International Conference Russe. Bulgarian, pp. 310-319.

Ibrahimov, V.R., Imanova, M.N. 2014. Hybrid methods for solving nonlinear ODE of the first order, Proceedings of the International Conference on Numerical Analysis and Applied Mathematics, (ICNAAM-2014) AIP Conf. Proc. 1648, © 2015 AIP Publishing LLC

Kobza, J. 1975. Second derivative methods of Adams type. Applikace Mathematicky, №20, p.389-405.

Makroglou, A. 1982. Hybrid methods in the numerical solution of Volterra integro-differential equations. *Journal of Numerical Analysis* 2, pp.21-35.

Mehdiyeva, G., Ibrahimov, V., Imanova, M. 2015. Solving Volterra Integro-Differential Equation by the Second Derivative Methods Applied Mathematics and Information Sciences, Volume 9, No. 5, Sep., 2521-2527

Mehdiyeva, G., Imanova, M., Ibrahimov, V. 2013. A way to construct an algorithm that uses hybrid methods. Applied Mathematical Sciences, HIKARI Ltd, Vol. 7, no. 98, p.4875-4890.

Mehdiyeva, G., Imanova, M., Ibrahimov, V. 2012. An application of the hybrid methods to the numerical solution of ordinary diferential equations of second order. Kazakh National University named after Al-Farabi *Journal of treasury series mathematics*, mechanics, computer science, Almaty №4 (75) p. 46-54.

Mehdiyeva, G. Yu., Nasirova, I.I., Ibrahimov, V.R. 2005. On some connections between Runge-Kutta and Adams methods Transactions issue mathematics and mechanics series of physical-technical and mathematical science, No5, P.55-62.

Srimani, P.K. and M.C. Roopa 2011. The cumulative effect of rotation on the onset of bio-porous-convection (bpc) in a suspension of gyrotactic microorganisms in a layer of finite depth under adverse temperature gradient *International Journal of Current Research Vol*. 3, Issue, 9, pp.114-119, August.

Sekar, S. and Kumar, 2011. Numerical investigation of nonlinear Volterra-Hammerstein integral equations via single-term haar wavelet series *International Journal of Current Research* Vol. 3, Issue, 2, pp.099-103, February.

Skvortsov, L.M. 2009. Explicit two-step Runge-Kutta methods Math. modeling, 21, 9, P. 54-65.

Subbotin, M.F. 1937. Kurs nebesnoy mekhaniki t.2, ONTI, Moskow, 1937, 404p.

****