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RESEARCH ARTICLE

INVERSION THEOREM FOR DISTRIBUTIONAL FOURIER-LAPLACE TRANSFORM

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ABSTRACT

The methods of integral transforms are very efficient to solve and research differential and integral equations of mathematical physics. These methods consist in the integration of an equation with some weight function of two arguments that often results in the simplification of a given initial problem. The main condition for the application of an integral transform is the validity of the inversion theorem which allows one to find an unknown function knowing its image. The aim of the present paper is to provide the generalization of Fourier-Laplace transform in the distributional sense and giving inversion theorem for the distributional Fourier-Laplace transform.

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1. INTRODUCTION

Fourier and Laplace Transforms continue to be a very important tool for the engineer, physicist and applied mathematician. They are also now useful to financial, economic and biological modelers as these disciplines become more quantitative. Communication is all based on Mathematics, be it digital, wired or wireless using Fourier Transform analysis. Integral transforms provide a way to solve otherwise intractable physical problems. They work by expressing the equations of a physical system in a new form that can be solved with simple computation. An example is the Laplace transform, which renders a useful class of differential equations trivially solvable, converting them into algebraic ones instead. Another is the very well-known Fourier transform, which maps functions of Cartesian coordinates to functions of frequencies (Berian J. James, 2008). These transforms play an important role in the analysis of all kinds of physical phenomenon. As a link between the various applications of these transforms the authors use the theory of signals and systems, as well as the theory of ordinary and partial differential equations (Beerends *et al*., 2003). The use of Fourier/Laplace transforms to evaluate numerically relevant probabilities in ruin theory as an application to insurance. The transform of a function is split in two; the real and imaginary parts and used an inversion formula based on the real parts only, to get the original function (Fatima *et al*., 2002). Numerical transform inversion has an odd place in computational probability. Historically, transforms have been exploited extensively for solving queuing and related probability models, but only rarely was numerical inversion attempted (Abate, Joseph *et al*., 1999). Many authors studied on various integral transforms separately. However there is much scope in extending double transformation to a certain class of generalized functions. Bhosale and Choudhary (2002); Khairnar *et al*. (2012) has discussed double transform. Motivated by this we have also defined a new combination of integral transforms in distributional generalized sense namely Fourier-Laplace transform. Along with the definition its analyticity theorem (Sharma and Rangari, 2011), Abelian theorem (Sharma and Rangari, 2014) and Representation theorem (Sharma and Rangari, 2014) are proved. Motivated by this, Inversion Theorem for distributional Fourier-Laplace transform is presented in this paper.

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The planning of this paper is as follows

Testing function spaces are described in section 2, In section 3, we have given the definition of Distributional generalized Fourier-Laplace transform. We have proved Inversion theorem for distributional Fourier-Laplace transform with two lemmas which are given in section 4. In section 5 Uniqueness theorem is given. Lastly conclusions are given by section 6. Notations and terminology are as per Zemanian (Zemanian, 1968; Zemanian, 1965).

2. Testing Function Spaces

2.1. The space $FL_{a,b,\alpha}$

This space is given by

$$FL_{a,b,\alpha} = \left\{ \phi : \phi \in E_+ / \xi_{a,b,k,q,l}\phi(t,x) = \sup_{\substack{0<t<\infty \\ 0<x<\infty}} \left| t^k e^{ax} D_t^l D_x^q \phi(t,x) \right| \le C_{lq} A^k k^{k\alpha} \right\} \quad (2.1)$$

Where the constants A and C_{lq} depend on the testing function ϕ.

2.2. The Space $FL_{a,b,\gamma}$

It is given by

$$FL_{a,b,\gamma} = \left\{ \phi : \phi \in E_+ / \gamma_{a,b,k,q,l}\phi(t,x) = \sup_{\substack{0<t<\infty \\ 0<x<\infty}} \left| t^k e^{ax} D_t^l D_x^q \phi(t,x) \right| \le C_{lk} A^q q^{q\gamma} \right\} \quad (2.2)$$

Where, $k,l,q = 0,1,2,3,........$ and the constants depend on the testing function ϕ.

3. Distributional Generalized Fourier-Laplace transforms (FLT)

For $f(t,x) \in FL_{a,\alpha}^{*\beta}$, where $FL_{a,\alpha}^{*\beta}$ is the dual space of $FL_{a,\alpha}^{\beta}$. It contains all distributions of compact support. The distributional Fourier-Laplace transform is a function of $f(t,x)$ and is defined as

$$FL\{f(t,x)\} = F(s,p) = \left\langle f(t,x), e^{-i(st-ipx)} \right\rangle, \quad (3.1)$$

where, for each fixed t $(0<t<\infty)$, x $(0<x<\infty)$, $s>0$ and $p>0$, the right hand side of (3.1) has a sense as an application of $f(t,x) \in FL_{a,\alpha}^{*\beta}$ to $e^{-i(st-ipx)} \in FL_{a,\alpha}^{\beta}$.

4. Inversion Theorem for Distributional Fourier-Laplace Transform

4.1 Lemma 1

Statement:-Let $FL\{f(t,x)\} = F(s,p)$ and $\sup f \subset S_A \cap S_B$, where $S_A = \{t : t \in R^n, |t| \le A, A>0\}$ and $S_B = \{x : x \in R^n, |x| \le B, B>0\}$, for $s>0$ and $\rho_1 < \operatorname{Re} p < \rho_2$. Let $\phi \in D$ and $\psi(s,p) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(t,x) e^{i(st-ipx)} dtdx$

(4.1.1)

Then for any fixed real number τ and r with $-\infty < r < \infty$, $-\infty < \tau < \infty$, $\int_{-r}^{r}\int_{-\tau}^{\tau} \left\langle f(t,x), e^{-i(st-ipx)} \right\rangle \psi(s,p) dsdw = \left\langle f(t,x), \int_{-r}^{r}\int_{-\tau}^{\tau} e^{-i(st-ipx)} \psi(s,p) dsdw \right\rangle$, where $p = \rho + iw$, also s and ρ are fixed with $\sigma_1 < s < \sigma_2$ and $\rho_1 < p < \rho_2$.

Proof: For $\phi(t,x) \equiv 0$, the result is trivial, so assume that $\phi(t,x) \neq 0$. If $FL\{f(t,x)\} = F(s,p)$, then $F(s,p)$ is analytic for $s > 0$, $\rho_1 < \operatorname{Re} p < \rho_2$ and $\psi(s,p)$ is an entire function. Therefore above integrals certainly exist.

In order that right hand side is meaningful, we show that

$$\int_{-r}^{r}\int_{-\tau}^{\tau} \psi(s,p) e^{-i(st-ipx)} dsdw \in FL_{a,b,\alpha}.$$

Consider,

$$\left| t^k K_{a,b}(x) D_t^l D_x^q \int_{-r}^{r}\int_{-\tau}^{\tau} e^{-i(st-ipx)} \psi(s,p)\, dsdw \right|$$

$$\leq \int_{-r}^{r}\int_{-\tau}^{\tau} \left| t^k K_{a,b}(x) D_t^l D_x^q e^{-i(st-ipx)} \psi(s,p) \right| dsdw$$

$$\leq \int_{-r}^{r}\int_{-\tau}^{\tau} \left| t^k K_{a,b}(x) (-i)^l s^l (-p)^q e^{-i(st-ipx)} \psi(s,p) \right| dsdw$$

$$\leq \int_{-r}^{r}\int_{-\tau}^{\tau} \left| t^k K_{a,b}(x) (s)^l (p)^q e^{-i(st-ipx)} \psi(s,p) \right| dsdw$$

$$\leq \int_{-r}^{r}\int_{-\tau}^{\tau} \left| t^k K_{a,b}(x) (s)^l (p)^q e^{-ist} e^{-px} \psi(s,p) \right| dsdw$$

$$\leq \int_{-r}^{r}\int_{-\tau}^{\tau} \left| t^k K_{a,b}(x) (s)^l (p)^q e^{-ist} e^{-(\rho+iw)x} \psi(s,p) \right| dsdw$$

$$\leq \int_{-r}^{r}\int_{-\tau}^{\tau} \left| t^k K_{a,b}(x) (s)^l (p)^q e^{-ist} e^{-\rho x} \psi(s,p) e^{-iwx} \right| dsdw < \infty \qquad (4.1.1)$$

$$\Rightarrow \int_{-r}^{r}\int_{-\tau}^{\tau} e^{-i(st-ipx)} \psi(s,p)\, dsdw \in FL_{a,b,\alpha}.$$

Partition the path of integration on the straight line from $s = -r$ to $s = r$ into m-intervals, each of length $\frac{2r}{m}$ and from $p = \rho - i\tau$ to $p = \rho + i\tau$ into n-intervals, each of length $\frac{2\tau}{n}$.

Let $S_\nu = \sigma$ be any point in ν^{th} interval and $p_\mu = \rho + iw$ be any point in μ^{th} interval.

Suppose, $$\phi_{m,n}(t,x) = \sum_{\nu=1}^{m}\sum_{\mu=1}^{n} e^{-i(s_\nu t - ip_\mu x)} \psi(s_\nu, p_\mu) \frac{2r}{m}\frac{2\tau}{n} \qquad (4.1.2)$$

To show that $\phi_{m,n}(t,x)$ converges in $FL_{a,b,\alpha}$ to $\int_{-r}^{r}\int_{-\tau}^{\tau} e^{-i(st-ipx)} \psi(s,p)\, dsdw$, we have to show that $\phi_{m,n}(t,x) - \int_{-r}^{r}\int_{-\tau}^{\tau} e^{-i(st-ipx)} \psi(s,p)\, dsdw$, converges to zero in $FL_{a,b,\alpha}$ as $m,n \to \infty$

We write,

$$\left|t^k K_{a,b}(x) D_t^l D_x^q \left[\phi_{m,n}(t,x) - \int_{-r}^{r}\int_{-\tau}^{\tau} e^{-i(st-ipx)}\psi(s,p)\,dsdw\right]\right|$$

$$= \left|t^k K_{a,b}(x) D_t^l D_x^q \left[\sum_{\nu=1}^{m}\sum_{\mu=1}^{n} e^{-i(s_\nu t - ip_\mu x)}\psi(s_\nu, p_\mu)\frac{2r}{m}\frac{2\tau}{n} - \int_{-r}^{r}\int_{-\tau}^{\tau} e^{-i(st-ipx)}\psi(s,p)\,dsdw\right]\right|$$

$$= \left|t^k K_{a,b}(x)(-i)^l(-1)^q \sum_{\nu=1}^{m}\sum_{\mu=1}^{n} s_\nu^l p_\mu^q e^{-i(s_\nu t - ip_\mu x)}\psi(s_\nu, p_\mu)\frac{2r}{m}\frac{2\tau}{n} - (-i)^l(-1)^q t^k K_{a,b}(x)\int_{-r}^{r}\int_{-\tau}^{\tau} s^l p^q e^{-i(st-ipx)}\psi(s,p)\,dsdw\right|$$

$$= \left|t^k K_{a,b}(x)\sum_{\nu=1}^{m}\sum_{\mu=1}^{n} s_\nu^l p_\mu^q e^{-i(s_\nu t - ip_\mu x)}\psi(s_\nu, p_\mu)\frac{2r}{m}\frac{2\tau}{n} - t^k K_{a,b}(x)\int_{-r}^{r}\int_{-\tau}^{\tau} s^l p^q e^{-i(st-ipx)}\psi(s,p)\,dsdw\right| \quad (4.1.3)$$

Since $\int_{-r}^{r}\int_{-\tau}^{\tau} s^l p^q \psi(s,p)\,dsdw$ is finite by (4.1.1) and $\left|t^k K_{a,b}(x) e^{-i(st-ipx)}\right| \to 0$, for sufficiently large values of x and t.

Given any $\in > 0$, we can choose x_0 and t_0 so large that for $x > x_0$ and $t > t_0$,

$$\left|t^k K_{a,b}(x)\int_{-r}^{r}\int_{-\tau}^{\tau} s^l p^q e^{-i(st-ipx)}\psi(s,p)\,dsdw\right| < \frac{\varepsilon}{3} \quad (4.1.4)$$

Now consider the first term of (4.1.3), choosing m_0 and n_0 so large that, for $m > m_0$ and $n > n_0$,

$$\left|t^k K_{a,b}(x)\sum_{\nu=1}^{m}\sum_{\mu=1}^{n} s_\nu^l p_\mu^q e^{-i(s_\nu t - ip_\mu x)}\psi(s_\nu, p_\mu)\frac{2r}{m}\frac{2\tau}{n}\right| < \frac{2\varepsilon}{3}, \text{ for all } x > x_0 \text{ and } t > t_0. \quad (4.1.5)$$

In view of above inequalities (4.1.4), (4.1.5) and (4.1.3),

$$\left|t^k K_{a,b}(x) D_t^l D_x^q \left[\phi_{m,n}(t,x) - \int_{-r}^{r}\int_{-\tau}^{\tau} e^{-i(st-ipx)}\psi(s,p)\,dsdw\right]\right| < \varepsilon$$

$\Rightarrow \phi_{m,n}(t,x)$ converges to $\int_{-r}^{r}\int_{-\tau}^{\tau} e^{-i(st-ipx)}\psi(s,p)\,dsdw$ in $FL_{a,b,\alpha}$.

Hence $$\left\langle f(t,x), \phi_{m,n}(t,x)\right\rangle = \left\langle f(t,x), \int_{-r}^{r}\int_{-\tau}^{\tau} e^{-i(st-ipx)}\psi(s,p)\,dsdw\right\rangle \quad (4.1.6)$$

Further left hand side of (4.1.6)

$$= \left\langle f(t,x), \sum_{\nu=1}^{m}\sum_{\mu=1}^{n} e^{-i(s_\nu t - ip_\mu x)}\psi(s_\nu, p_\mu)\frac{2r}{m}\frac{2\tau}{n}\right\rangle$$

$$= \sum_{\nu=1}^{m}\sum_{\mu=1}^{n}\left\langle f(t,x), e^{-i(s_\nu t - ip_\mu x)}\psi(s_\nu, p_\mu)\frac{2r}{m}\frac{2\tau}{n}\right\rangle$$

$$= \sum_{\nu=1}^{m}\sum_{\mu=1}^{n}\left\langle f(t,x), e^{-i(s_\nu t - ip_\mu x)}\right\rangle\psi(s_\nu, p_\mu)\frac{2r}{m}\frac{2\tau}{n}$$

$$= \int_{-r}^{r}\int_{-\tau}^{\tau}\left\langle f(t,x), e^{-i(st-ipx)}\right\rangle\psi(s,p)\,dsdw.$$

Since $\left\langle f(t,x), e^{-i(s_\nu t - ip_\mu x)} \right\rangle \psi(s_\nu, p_\mu)$ is a continuous function of s and w from (4.1.6). We have

$$\int_{-r}^{r}\int_{-\tau}^{\tau} \left\langle f(t,x), e^{-i(st-ipx)} \right\rangle \psi(s,p)\, ds dw = \left\langle f(t,x), \int_{-r}^{r}\int_{-\tau}^{\tau} e^{-i(st-ipx)} \psi(s,p)\, ds dw \right\rangle.$$

4.2. Lemma 2

Let a,b,c,d,ρ,r and τ be real numbers with $c<s<d, a<p<b$. Let $\phi \in D$ then $\frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(t,x)\frac{\sin(t-v)r}{t-v}\frac{\sinh(x-y)\tau}{x-y}\, dtdx = A(v,y)$ converges in $FL_{a,b,\alpha}$ to $\phi(v,y)$ as $r\to\infty, \tau\to\infty$.

Proof: To prove

$$\frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(t,x)\frac{\sin(t-v)r}{(t-v)}.\frac{\sinh(x-y)\tau}{(x-y)}\, dtdx \to \phi(v,y)$$

We have to show that $\xi_{l,q}\left|A(v,y)-\phi(v,y)\right| \to 0$, where $\xi_{l,q}\left|A(v,y)\right| = S\,\text{up}\left|v^k e^{ay} D_v^l D_y^q A(v,y)\right|$

Consider,

$$\xi_{l,q}\left|A(v,y)-\phi(v,y)\right|$$
$$= Sup\left| v^k e^{ay} D_v^l D_y^q \left\{ \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(t,x)\frac{\sin(t-v)r}{(t-v)}.\frac{\sinh(x-y)\tau}{(x-y)}\, dtdx - \phi(v,y) \right\} \right| \to 0 \qquad (4.2.1)$$

Because,

$$\int_{-\infty}^{\infty} \frac{\sin(t-v)r}{(t-v)}\, dt = \int_{-\infty}^{\infty} \frac{\sin rt_1}{t_1}\, dt_1 = \pi$$

$$\int_{-\infty}^{\infty} \frac{\sinh(x-y)\tau}{(x-y)}\, dx = \int_{-\infty}^{\infty} \frac{\sinh \tau x_1}{x_1}\, dx_1 = \pi$$, where $r>0$, $\tau>0$

So, $$\frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(v,y).\frac{\sin(t-v)r}{(t-v)}.\frac{\sinh(x-y)\tau}{(x-y)}\, dtdx = \phi(v,y) \qquad (4.2.2)$$

In view of (4.2.1) and (4.2.2), we have to prove

$$\sup\left| v^k e^{ay} D_v^l D_y^q \left[\frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(t,x)\frac{\sin(t-v)r}{(t-v)}\frac{\sinh(x-y)\tau}{(x-y)}\, dtdx - \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(v,y)\frac{\sin(t-v)r}{(t-v)}\frac{\sinh(x-y)\tau}{(x-y)}\, dtdx \right]\right|$$
$$= \left| v^k e^{ay} D_v^l D_y^q \left[\frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{\phi(t,x)-\phi(v,y)\right\}\frac{\sin(t-v)r}{(t-v)}\frac{\sinh(x-y)\tau}{(x-y)}\, dtdx \right]\right|$$
$$= \left| v^k e^{ay} D_v^l D_y^q \left[\frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{\phi(t_1+v, x_1+y)-\phi(v,y)\right\}\frac{\sin(t_1 r)}{t_1}\frac{\sinh x_1\tau}{x_1}\, dt_1 dx_1 \right]\right| \because r \text{ and } \tau > 0$$

$$\{t-v=t_1 \therefore t=t_1+v$$
$$x-y=x_1 \therefore x=x_1+y$$

Converges to zero, as in Zemanian (1965) pp.66 Theorem 3.5.1.

Hence the theorem is proved.

Now we prove the Theorem.

4.3. Inversion Theorem

Statement: - Let $FL\{f(t,x)\}=F(s,p)$, for $s>0$ and $\rho_1<p<\rho_2$. Also, let r and τ be a real variables such that $-\infty<r<\infty$, $-\infty<\tau<\infty$. Then in the sense of convergence in D^*, $f(t,x)=\lim_{\substack{r\to\infty\\ \tau\to\infty}} \frac{1}{4\pi^2}\int_{-r}^{r}\int_{-\tau}^{\tau} F(s,p)e^{i(st-ipx)}dsdp$, where s and p are fixed real numbers with $-r<s<r$ and $\rho_1<p<\rho_2$.

Proof: - Let $\phi\in D$. Choose the real numbers c and d such that $c<s<d$ and the real numbers a and b such that $\rho_1<a<p<b<\rho_2$, we have to show that $\langle f,\phi\rangle = \lim_{\substack{r\to\infty\\ \tau\to\infty}}\left\langle \frac{1}{4\pi^2}\int_{-r}^{r}\int_{-\tau}^{\tau} F(s,p)e^{i(st-ipx)}dsdp,\phi(t,x)\right\rangle$

(4.3.1)

Now, the integral on s and p is a continuous function of t and x and therefore the right hand side of (4.3.1) without the limit notation can be written as

$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{0}^{\infty}\phi(t,x)\int_{-r}^{r}\int_{-\tau}^{\tau} F(s,p)e^{i(st-ipx)}dsdpdtdx,\quad r,\tau>0 \tag{4.3.2}$$

Since $\phi(t,x)$ is of bounded support and the integrand is a continuous function of t,x,s,p, the order of integration may be changed and we write $\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{0}^{\infty}\phi(t,x)\int_{-r}^{r}\int_{-\tau}^{\tau} F(s,p)e^{i(st-ipx)}dsdpdtdx = \frac{1}{4\pi^2}\int_{-r}^{r}\int_{-\tau}^{\tau}F(s,p)\int_{-\infty}^{\infty}\int_{0}^{\infty}\phi(t,x)e^{i(st-ipx)}dtdxdsdp$

$$=\frac{1}{4\pi^2}\int_{-r}^{r}\int_{-\tau}^{\tau}\left\langle f(v,y),e^{-i(sv-ipy)}\right\rangle\int_{-\infty}^{\infty}\int_{0}^{\infty}\phi(t,x)e^{i(st-ipx)}dtdxdsdp$$

$$=\left\langle f(v,y),\frac{1}{4\pi^2}\int_{-r}^{r}\int_{-\tau}^{\tau}e^{-i(sv-ipy)}\int_{-\infty}^{\infty}\int_{0}^{\infty}\phi(t,x)e^{i(st-ipx)}dtdxdsdp\right\rangle \quad \text{(By Lemma 1)}$$

The order of integration for the repeated integral herein may be changed because again $\phi(t,x)$ is of bounded support and the integrand is a continuous function of t,x,s,p upon doing this we obtain

$$=\left\langle f(v,y),\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{0}^{\infty}\phi(t,x)\int_{-r}^{r}\int_{-\tau}^{\tau}e^{-isv}e^{-py}e^{ist}e^{px}dtdxdsdp\right\rangle$$

$$=\left\langle f(v,y),\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{0}^{\infty}\phi(t,x)\int_{-r}^{r}\int_{-\tau}^{\tau}e^{(t-v)is}e^{p(x-y)}dtdxdsdp\right\rangle$$

$$=\left\langle f(v,y),\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{0}^{\infty}\phi(t,x)\left[\int_{-r}^{r}e^{(t-v)is}ds\right]\left[\int_{-\tau}^{\tau}e^{p(x-y)}dp\right]dtdx\right\rangle$$

$$=\left\langle f(v,y),\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{0}^{\infty}\phi(t,x)\frac{2\sin(t-v)r}{(t-v)}\frac{2\sinh(x-y)\tau}{(x-y)}dtdx\right\rangle$$

$$=\left\langle f(v,y),\frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{0}^{\infty}\phi(t,x)\frac{\sin(t-v)r}{(t-v)}\frac{\sinh(x-y)\tau}{(x-y)}dtdx\right\rangle$$

Taking $r\to\infty,\tau\to\infty$ and using Lemma 2, we get

$$=\langle f(v,y),\phi(v,y)\rangle$$

$$\therefore \lim_{\substack{r\to\infty\\ \tau\to\infty}}\left\langle\frac{1}{4\pi^2}\int_{-r}^{r}\int_{-\tau}^{\tau}F(s,p)e^{i(st-ipx)}dsdp,\phi(t,x)\right\rangle=\langle f,\phi\rangle$$

This completes the proof.

5. Uniqueness Theorem

If $FL\{f(t,x)\}=F(s,p)$, for $s,p\in\Omega_f$ and $FL\{g(t,x)\}=G(s,p)$, for $s,p\in\Omega_g, s>0$ and $\rho_1<\operatorname{Re}p<\rho_2$. If $\Omega_f\cap\Omega_g$ is not empty and if $F(s,p)=G(s,p)$, for $s\in\Omega_f\cap\Omega_g$ and $p\in\Omega_f\cap\Omega_g$ then $f=g$ in the sense if equality $D^*(I)$.

Proof

f and g must assign the same value to each $\phi\in D$. By inversion theorem and equating $F(s,p)$ and $G(s,p)$ in $\langle f-g,\phi(t,x)\rangle$

$$=\lim_{\substack{r\to\infty\\ \tau\to\infty}}\left\langle\frac{1}{4\pi^2}\int_{-r}^{r}\int_{-\tau}^{\tau}(F-G)(s,p)e^{i(st-ipx)}dsdp,\phi(t,x)\right\rangle=0$$

Thus, $f=g$ in $D^*(I)$.

6. Conclusion

This paper proved Inversion Theorem for Distributional Fourier-Laplace Transform with the help of two lemmas. Also proved Uniqueness theorem.

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