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RESEARCH ARTICLE

STATISTICAL ASPECTS OF BOSONS AND FERMIONS

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ABSTRACT

Attempts on the occupation of particles (Frederick Reif., 2010; K.Huang. 2009: Mac Donald, 2006; R.K. Srivastava and J Ashok 2005; Avijith Lahiri, 2013; Landau and Lifshitz, 1969) relating to Bosons and Fermions that are indistinguishable particles is still attracting attention in comparison to ideal gas. These particles possess certain special characteristics relevant to its spin, indistinguishability, wave function, symmetry and its restriction. Sequential procedure is followed in obtaining equation for Bose gas and Fermi gas and comparison is implicated with ideal gas.

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INTRODUCTION

The attempt in occupations of these particles are formulated separately for Ideal Bose Gas and Ideal Fermi Gas

Ideal Bose gas

Bose gas containing bosons is realized, computed energies and number of particles on approximation leads to ideal gas. Special characteristics of these particles (Bosons) typically include are (i) indistinguishable (ii)exhibit no change in sign of the wave function with interchange of particles (iii) No restriction to the number of particles occupying a given state (iv) characterized by integral spin termed as Bosons. Such a realization of Bose gas with special kind of particles were initiated by S.N. Bose and further extended by Einstein. The occupation of these particles in various quantum states g_i reveal its population n_i corresponding to i^{th} state characterized with energy $£_i$ .The total number of particles corresponding to whole Bose gas of r quantum states of bosons is $\sum_r n_r = N$ and total energy $\sum_r n_r £_r = E$. Particles like bosons are photons, liquid helium, gluons, etc.

No. of particles in a range $£$ to $£ + d£$ corresponding to i^{th} quantum state is

$$n(£)d£ = \frac{g_i(£_i)d£_i}{e^{~+ £_i} - 1}$$

$$n(£ \ )d£ = \frac{g_i(£_i)d£_i}{e^{(£_i - «)} - 1}$$

Where $~ = -$ «, $= \frac{1}{kT}, g_i$ is the degeneracy of i^{th} state corresponding to energy $£_i$, « is chemical potential, k is Boltzmann

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constant and T is absolute temperature. For B.E statistics the total number of particles summed over all the possible states is N *i.e* $\sum_r n_r = N$. As the number of particles in a state must be greater than zero(cannot be zero) $« \leq 0$ $\exp(\frac{-«}{kT}) \geq 1$. The kinetic energy of particle is $£ = \frac{p^2}{2m}$.The number of quantum states with volume V whose momentum lies between p and $p+dp$ is $g(p)dp = \frac{4®Vp^2dp}{h^3}$ with a total of $2s+1$ spin values. On generalization to total states of whole gas we transform from momentum to energy variables as these related to kinetic energy

To determine $g(£)d£$ with $£ = \frac{p^2}{2m} \Rightarrow p^2 = 2m£$ and $p = \sqrt{2m£}$ $dp = \frac{d£}{\sqrt{2m£}}$

Hence $p^2dp = (2m)^{3/2}£^{1/2}d£\frac{1}{2}$

$$g(£)d£ = \frac{2®gV}{h^3}(2m)^{3/2}£^{1/2}d£$$

Hence the total number of particles is $N = \sum_r n_r = \int_0^\infty \frac{g(£)d£}{e^{~+\ £}-1}$

$$N = \frac{2®gV}{h^3}(2m)^{3/2}\int_0^\infty £^{1/2}\frac{d£}{e^{\ (£-«)}-1}$$

As per the distribution the total number of particles in terms of mean number $N = \tilde{n}_i$ of particles and their occupation is $g(£)d£$ with energy $E = \int_0^\infty £n(£)d£$ i.e $E = \int_0^\infty \frac{£g(£)}{e^{\ £}Z^{-1}-1}$

$$E = \frac{2®gV}{h^3}(2m)^{3/2}\int_0^\infty \frac{£^{\ 3/2}}{e^{\ (£-«)}-1}d£$$

With proper substitutions of $\frac{£}{kT} = x$; $d£ = (kT)dx$

$$N = \frac{2®gV}{h^3}(2m)^{3/2}\int_0^\infty \frac{x^{1/2}(kT)^{1/2}dx(kT)}{e^x e^{-\ «}-1}$$

$$N = \frac{2®gV}{h^3}(2mkT)^{3/2}\int_0^\infty \frac{x^{1/2}dx}{e^x e^{-\ «}-1}$$

With substitution of $Z^{-1} = e^{-\ «}$ $Z = e^{\ «}$

$$N = \frac{2®gV}{h^3}(2mkT)^{3/2}\int_0^\infty \frac{x^{1/2}dx}{e^x Z^{-1}-1}$$

$$E = \frac{2®gV}{h^3}(2m)^{3/2}\int_0^\infty \frac{x^{3/2}(kT)^{3/2}dx(kT)}{e^x Z^{-1}-1}$$

$$E = \frac{2®gV}{}\left(\frac{2mkT}{h^2}\right)^{3/2}(kT)\int_0^\infty \frac{x^{3/2}dx}{e^x Z^{-1} - 1}$$

$Z = e^{«} << 1$ is the limiting case of Boltzmann distribution.

Evaluation of integral in computation of E and N requires solving term in denominator

$$[Z^{-1}e^x - 1]^{-1} = \frac{1}{Z^{-1}e^x\left(1 - Ze^{-x}\right)} = Ze^x\left(1 - Ze^{-x}\right)^{-1} = Ze^x\left(1 + Ze^{-x}\right)$$

Hence

$$N = \frac{2®gV}{h^3}(2mkT)^{3/2}\int_0^\infty \frac{x^{1/2}dx}{e^x Z^{-1} - 1}$$

$$N = 2®gV\left(\frac{2mkT}{h^2}\right)^{3/2}\int_0^\infty x^{1/2}dxZe^x\left(1 + Ze^{-x}\right)$$

Considering the integral part $\int_0^\infty Ze^{-x}x^{1/2}dx + \int_0^\infty x^{1/2}Z^2e^{-2x}dx$

$$Z\int_0^\infty e^{-x}x^{1/2}dx + Z^2\int_0^\infty x^{1/2}e^{-2x}dx$$

$Z\int_0^\infty e^{-x}x^{(\frac{3}{2}-1)}dx + Z^2\int_0^\infty x^{1/2}e^{-2x}dx$ with transformation from variable x to variable u Integral transforms to

$$Z\int_0^\infty e^{-x}x^{(\frac{3}{2}-1)}dx + Z^2\int_0^\infty \left(\frac{u}{2}\right)^{1/2} e^{-u}\,\frac{du}{2}$$

$$Z\Gamma_{3/2} + \frac{Z^2}{2^{3/2}}\int_0^\infty u^{1/2}e^{-u}du$$

$$Z\frac{1}{2}\Gamma_{1/2} + \frac{Z^2}{2^{3/2}}\int_0^\infty u^{\left(\frac{3}{2}-1\right)}e^{-u}du$$

$$Z\frac{\sqrt{®}}{2} + \frac{Z^2}{2^{3/2}}\frac{\sqrt{®}}{2} = \sqrt{®}\left(\frac{Z}{2} + \frac{Z^2}{2^{5/2}}\right)$$

Hence the number of particles is

$$N = 2®gV\left(\frac{2mkT}{h^2}\right)^{3/2}\frac{\sqrt{®}}{2}\left(Z + \frac{Z^2}{2\sqrt{2}}\right)$$

$$N = 2®gV\left(\frac{2mkT}{h^2}\right)^{3/2}\frac{\sqrt{®}}{2}Z\left(1 + \frac{Z}{2\sqrt{2}}\right)$$

Solving for Z

$$Z=\frac{N}{gV}\left(\frac{h^2}{2®mkT}\right)^{3/2}\left(1+\frac{Z}{2\sqrt{2}}\right)^{-1}$$

$$Z=\frac{N}{gV}\left(\frac{h^2}{2®mkT}\right)^{3/2}\left[1-\frac{1}{2\sqrt{2}}\left(\left(\frac{h^2}{2®mkT}\right)^{3/2}\frac{N}{gV}\left(1+\frac{Z}{2\sqrt{2}}\right)^{-1}\right)\right]$$

$$Z=\frac{N}{gV}\left(\frac{h^2}{2®mkT}\right)^{3/2}\left[1-\frac{1}{2\sqrt{2}}\frac{N}{gV}\left(\frac{h^2}{2®mkT}\right)^{3/2}+...........\right]$$

Similarly evaluation of

$$\int_0^\infty\frac{x^{3/2}dx}{e^{-x}e^{-«}-1}=\frac{3}{4}\sqrt{®}\,Z\left(1+\frac{Z}{\sqrt{2}}\right)$$

Hence

$$E=\frac{3}{2}gVkT\left(\frac{2m®kT}{h^2}\right)^{3/2}\left[\frac{N}{gV}\left(\left(\frac{h^2}{2m®kT}\right)^{3/2}\left(1-\frac{1}{2\sqrt{2}}\frac{N}{gV}\left(\frac{h^2}{2m®kT}\right)^{3/2}+.........\right)\right)\right]$$

$$E=\frac{3}{2}NkT$$

First term is known as the Boltzmann term and other term is the correction to energy

$PV=NkT\left[1-\frac{1}{4\sqrt{2}}\frac{N}{gV}\left(\frac{h^2}{2®mkT}\right)^{3/2}-..........\right]$ is the equation of Ideal Bose gas

It resembles the ideal gas equation $PV=NkT$ when $\left[\frac{N}{gV}\left(\frac{h^2}{2®mkT}\right)^{3/2}\right]$ <<< 1

Ideal Fermi gas

Fermi gas containing fermions is realized, computed energies and number of particles on approximation leads to ideal gas. Special characteristics of these particles (fermions) typically include are these particles (i) are indistinguishable(ii)exhibit change in sign of the wave function with interchange of particles(iii) with restriction to the number of particles occupying a given state is either 0 or 1 (iv) characterized by half integral spin termed as fermions. Such a realization of Fermi gas with special kind of particles were initiated by Fermi and further extended by Dirac. The occupation of these particles in various quantum states g_i reveal its population n_i corresponding to i^{th} state characterized with energy $£_i$ .The total number of particles corresponding to whole Fermi gas of fermions is $\sum_r n_r=N$ and total energy $\sum_r n_r£_r=E$ Particles like fermions are electrons, protons, mesons, positron, etc.

No. of particles in a range $£$ to $£+d£$ corresponding to i^{th} quantum state is

$$n(£)d£=\frac{g_i(£_i)d£}{e^{~+£_i}+1}$$

$$n(£)d£=\frac{g_i(£_i)d£}{e^{(£_i-«)}+1}$$

$$N = \sum_r n_r = \int_0^\infty \frac{g(£)d£}{e^{\sim + \;£_i} + 1}$$

And total energy of whole gas is $E = \sum_r n_r £_r = \int_0^\infty \frac{£g(£)}{e^{\;(£-«)} + 1}$

Where $\sim = - \;«$, $\;= \frac{1}{kT}$, g_i is the degeneracy of i^{th} state corresponding to energy $£_i$, and « is chemical potential k is Boltzmann constant and T is absolute temperature.

The kinetic energy of particle is $£ = \frac{p^2}{2m}$ .The number of quantum states with volume V whose momentum lies between p and $p + dp$ is $g(p)dp = \frac{4®Vp^2 dp}{h^3}$ with a total of $2s+1$ spin values. Transforming from momentum to energy variables as these related to kinetic energy

With substitution of $Z^{-1} = e^{-\;«}$ $\;Z = e^{\;«}$

$$N = \int_0^\infty \frac{g(£)d£}{e^{\;£} Z^{-1} + 1} \text{ and } E = \int_0^\infty \frac{£g(£)}{e^{\;£} Z^{-1} + 1}$$

For F.D statistics the total number of particles summed over all the possible states is N i.e $\sum_r n_r = 0{,}1$

On generalization to total states of whole gas we transform from momentum to energy variables as these related to kinetic energy

To determine $g(£)d£$ with $£ = \frac{p^2}{2m} \Rightarrow p^2 = 2m£$ and $p = \sqrt{2m£}$ $\;dp = \frac{d£}{\sqrt{2m£}}$

Hence $p^2 dp = (2m)^{3/2} £^{1/2} d£ \frac{1}{2}$

$$g(£)d£ = \frac{2®gV}{h^3}(2m)^{3/2} £^{1/2} d£$$

Hence $N = \frac{2®gV}{h^3}(2m)^{3/2} \int_0^\infty \frac{£^{1/2} d£}{e^{\;£} Z^{-1} + 1}$ and $E = \frac{2®gV}{h^3}(2m)^{3/2} \int_0^\infty \frac{£^{3/2} d£}{e^{\;£} Z^{-1} + 1}$

Let define a function $F_{3/2} = \int_0^\infty \frac{£^{1/2} d£}{e^{\;£} Z^{-1} + 1}$ and $F_{5/2} = \int_0^\infty \frac{£^{3/2} d£}{e^{\;£} Z^{-1} + 1}$

Hence the total number of particles N and energy E transforms to

$$N = \frac{2®gV}{h^3}(2m)^{3/2} F_{3/2}$$

$$E = \frac{2®gV}{h^3}(2m)^{3/2} F_{5/2}$$

Defining in general $F_n(Z) = \int_0^\infty \frac{£^{n-1} d£}{e^{£} Z^{-1} + 1}$

With transformation from variable £ to x

$$\frac{£}{kT} = x \quad \text{or} \quad £ = x \quad \text{and} \quad d£ = (kT)dx$$

$$F_n(Z) = (kT)^n \int_0^\infty \frac{x^{n-1}dx}{e^x Z^{-1} + 1} \quad \text{and} \quad F_n(Z) = (kT)^n f_n(Z) \quad \text{with} \quad f_n(Z) = \frac{1}{\Gamma_n} \int_0^\infty \frac{x^{n-1}dx}{e^x Z^{-1} + 1}$$

Where Γ_n is gamma function and thermal wavelength $a = \sqrt{\frac{h^2}{2®mkT}}$

$$N = 2®gV\left(\frac{2m}{h^2}\right)^{3/2} \int_0^\infty \frac{£^{1/2} d£}{e^{£} Z^{-1} + 1}$$

$$\frac{£}{kT} = x \quad \text{or} \quad £ = x \quad £ = (kT)x \quad \text{and} \quad d£ = (kT)dx$$

$$N = 2®gV\left(\frac{2m}{h^2}\right)^{3/2} \int_0^\infty \frac{x^{1/2}dx}{{}^{1/2} \quad e^x Z^{-1} + 1}$$

$$N = 2®gV\left(\frac{2m}{h^2}\right)^{3/2} \frac{1}{{}^{3/2}} \int_0^\infty \frac{x^{1/2}dx}{e^x Z^{-1} + 1}$$

$$N = 2®gV\left(\frac{2m}{h^2}\right)^{3/2} \int_0^\infty \frac{x^{1/2}dx}{e^x Z^{-1} + 1}$$

$$N = 2®gV\left(\frac{2mkT}{h^2}\right)^{3/2} \int_0^\infty \frac{x^{1/2}dx}{e^x Z^{-1} + 1}$$

$$N = 2®gV\left(\frac{2mkT}{h^2}\right)^{3/2} \int_0^\infty \frac{x^{1/2}dx}{e^x Z^{-1} + 1}$$

$$N = 2®gV\left(\frac{2mkT}{h^2}\right)^{3/2} f_{3/2}(Z)\Gamma_{3/2}$$

$$N = 2®gV\left(\frac{2mkT}{h^2}\right)^{3/2} f_{3/2}(Z)\frac{1}{2}\sqrt{®}$$

$$N = gV\left(\frac{2®mkT}{h^2}\right)^{3/2} f_{3/2}(Z)$$

$$N = \frac{gV}{a^3} f_{3/2}(Z)$$

Similarly

$$E = \frac{2®gV}{h^3}(2m)^{3/2}\int_0^\infty \frac{£^{3/2}d£}{Z^{-1}e^{£}+1}$$

$£ = x$ Differentiating $d£ = dx$ and $d£ = \frac{dx}{}$

$$E = \frac{2®gV}{h^3}(2m)^{3/2}\frac{1}{.^{3/2}}\int_0^\infty \frac{x^{3/2}dx}{Z^{-1}e^{£}+1}$$

$$E = 2®gV\left(\frac{2mkT}{h^2}\right)^{3/2} kT\int_0^\infty \frac{x^{(5/2)-1}dx}{Z^{-1}e^{£}+1}$$

$$E = 2®gV\left(\frac{2mkT}{h^2}\right)^{3/2}(kT)\Gamma_{5/2K} f_{5/2}(Z)$$

$$E = 2®gV\left(\frac{2mkT}{h^2}\right)^{3/2}(kT)\frac{3}{2}\frac{1}{2}\sqrt{®}\, f_{5/2}(Z)$$

$$E = \frac{3}{2}kT\left(\frac{2m®kT}{h^2}\right)^{3/2} gVf_{5/2}(Z)$$

In terms of thermal wavelength

$$E = \frac{3}{2}kT\frac{gV}{a^3} f_{5/2}(Z)$$

To find values of $f_{3/2}$ and $f_{5/2}$

We know $f_n(Z) = \frac{1}{\Gamma_n}\int_0^\infty \frac{x^{n-1}dx}{Z^{-1}e^x+1}$

Denominator can be expressed as

$$\frac{1}{Z^{-1}e^x+1} = Ze^{-x}\left(1+Ze^{-x}\right)^{-1} = Ze^{-x}\left(1-\left(Ze^{-x}\right)+\left(Ze^{-x}\right)^2-........\right) = \sum_{l=0}^\infty Z^l e^{-lx}(-1)^{l-1}$$

Hence $f_n(Z) = \frac{1}{\Gamma_n}\int_0^\infty x^{n-1}dx\sum_{l=1}^\infty Z^l e^{-lx}(-1)^{l-1}$

$f_n(Z) = (-1)^{l-1}\frac{Z^l}{l^n}$ by definition of gamma function $\int_0^\infty e^{-x}x^{n-1}dx = \Gamma_n$

$$f_n(Z) = Z - \frac{Z^2}{2^n} + \frac{Z^3}{3^n} - \frac{Z^4}{4^n} + ...$$

Hence with $n = \frac{3}{2}$

$$f_{3/2}(Z) = Z - \frac{Z^2}{2^{3/2}} + \frac{Z^3}{3^{3/2}} - \frac{Z^4}{4^{3/2}} + ...$$

Now $N = \frac{gV}{a^3} f_{3/2}(Z)$ and $\frac{N}{V} = \frac{g}{a^3} f_{3/2}(Z)$ and $f_{3/2}(Z) = \frac{°}{g} a^3$ where $\frac{N}{V} = °$ is particle density

Case 1

At high temperature and low density $f_{3/2}(Z) << 1$ and gas is non degenerate $\Rightarrow Z << 1$

For Z small compared to unity $E = \frac{3}{2} kT \frac{gV}{a^3} f_{5/2}(Z)$ and $N = \frac{gV}{a^3} f_{3/2}(Z)$

$$E = \frac{3}{2} NkT \frac{f_{5/2}(Z)}{f_{3/2}(Z)}$$

With expansion of $f_n(Z) = Z - \frac{Z^2}{2^n} + \frac{Z^3}{3^n} - \frac{Z^4}{4^n} + ...$

$f_n(Z) = Z\left(1 - \frac{Z}{2^n} + \frac{Z^2}{3^n} - \frac{Z^3}{4^n} + ...\right)$ with substitution of $n = \frac{5}{2} or \frac{3}{2}$

hence $f_{3/2}(Z) = f_{5/2} \approx 1$

Hence total energy of systems is $E = \frac{3}{2} kT \frac{gV}{a^3} Z$ with inclusion of $N = \frac{gV}{a^3} Z$

Then total energy of system is $E = \frac{3}{2} NkT$ is result of classical ideal gas

Case 2

At low temperature and high density $\frac{°}{g} a^3 >> 1$ or $\frac{°}{g} a^3 \Rightarrow \infty$ .When $T \to 0$ the gas is degenerate.

For Fermi gas $\frac{Na^3}{Vg} >> 1$ with mean number of single particle sate with energy £

$\langle n_£ \rangle = \frac{1}{e^{~(£-«)} + 1}$ where « is chemical potential and ~ = − « and $~ = \frac{-«}{kT}$

No of quantum states with energy lying between £ to $£ + d£$ is

$$g(£)d£ = \frac{2®gV}{h^3}(2m)^{3/2} £^{1/2} d£$$

Hence the total number of particles in the range £ to $£ + d£$ is $n(£)d£ = \frac{g\ (£)d£}{e^{~(£-«)} + 1}$

$$n(£)d£ = \frac{2®gV}{h^3}(2m)^{3/2} \frac{£^{1/2} d£}{e^{~(£-«)} + 1}$$

Hence total number of particles is $N = \frac{2®gV}{h^3}(2m)^{3/2} \int_0^\infty \frac{£^{1/2} d£}{e^{~(£-«)} + 1}$ and

$$E = \frac{2 ® g V}{h^3}(2m)^{3/2}\int_0^\infty \frac{£^{3/2} d£}{e^{(£-«)}+1}$$

In the limit $T \rightarrow 0$ chemical potential $« \rightarrow «_o$ with completely degenerate and mean occupation number is $\langle n_£ \rangle = 1$ for $£ < «_0$ and $\langle n_£ \rangle = 0$ for $£ > «_0$. The limiting chemical potential $«_0$ corresponds to Fermi energy $£_f$ of the system. At $T = 0$ all the particles upto states $£ = £_f$ are filled with one particle in each state and all the particles with energy $£ > £_f$ are empty.

Conclusion

Equation of Ideal Bose gas and Fermi gas equations were obtained. These equations resemble with Ideal gas at high temperature for both Bose and Fermi gases. Particles though indistinguishable are realized with statistical aspects i.e. their occupation or distribution among various energy levels with their correspondence implicated in its mathematical form.

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