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RESEARCH ARTICLE

ON CON-K-NORMAL MATRICES

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ABSTRACT

The concept of conjugate k-normal (con-k-normal) matrices is introduced. Some basic results of con-k-normal, con-k-unitary are discussed.

INTRODUCTION

Let $\mathbb{C}_{nxn}$ be the space of nxn complex matrices. For a matrix $A \in \mathbb{C}_{n\times n}$, $\overline{A}, A^T, A^*, A^{-1}$ and $A^\dagger$ denote conjugate, transpose, conjugate transpose, inverse and Moore-Penrose inverse of a matrix 'A' respectively. Let 'k' be a fixed product of disjoint transpositions in S_n = {1, 2… n} (hence, involutory) and let 'K' be the associated permutation matrix. The concept of Con-k-normal matrices is introduced as a generalization of k-real and k-hermitian and normal matrices [2]. The con-k-unitary is also discussed in this paper. Clearly 'K' satisfies the following properties: $K^2 = I$ and $K = K^T = K^* = K^\dagger$.

Basic Definitions

Definition 2.1[3]: A matrix $A \in \mathbb{C}_{n\times n}$ is said to be k-normal, if $AA^*K = KA^*A$.

That is, $a_{ij}\,\overline{a}_{n-k(j)+1\;k(i)} = \overline{a}_{k(j)\;n-k(i)+1}\,a_{ij}$; i, j = 1, 2… n.

Definition 2.2[3]: A matrix $A \in \mathbb{C}_{n\times n}$ is said to be k-unitary, if $AA^*K = KA^*A = K$.

Con-k-normal matrices

Definition 3.1: A matrix $A \in \mathbb{C}_{n\times n}$ is said to be con-k-normal, if $AA^*K = \overline{KA^*A}$.

That is, $AA^*K = KA^T\overline{A}$ (or) $KA^*A = \overline{A}A^TK$

Example3.2: Let $A = \begin{bmatrix} i & 0 \\ 1 & i \end{bmatrix}$ is con-k-normal for k= (1, 2), where $K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Theorem 3.3: For $A \in \mathbb{C}_{n\times n}$ the following conditions are equivalent.

(i) A is con-k-normal.
(ii) $\overline{A}$ is con-k-normal.
(iii) A^T is con-k-normal.
(iv) A^* is con-k-normal.
(v) A^{-1} is con-k-normal, if A^{-1} exist.
(vi) $A^\dagger$ is con-k-normal.
(vii) λA is con-k-normal, where λ is a real number.

Proof: (i) $\Leftrightarrow$ (ii): A is con-k-normal

$$\Leftrightarrow AA^*K = KA^T\overline{A}$$

$$\Leftrightarrow \overline{(AA^*K)} = \overline{(KA^T\overline{A})}$$

$$\Leftrightarrow \overline{A}A^TK = KA^*A$$

$$\Leftrightarrow \overline{A} \text{ is con-k-normal.}$$

(i) $\Leftrightarrow$ (iii): A is con-k-normal

$$\Leftrightarrow AA^*K = KA^T\overline{A} \Leftrightarrow (AA^*K)^T = (KA^T\overline{A})^T$$

$$\Leftrightarrow K^T(A^*)^TA^T = \overline{A}^T(A^T)^TK^T \Leftrightarrow K\overline{A}A^T = A^*AK$$

Pre and post multiply by K on both sides,

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$\Leftrightarrow \overline{A}A^T K = K A^* A \Leftrightarrow A^T$ is con-k-normal.

(i) $\Leftrightarrow$ (iv): A is con-k-normal $\Leftrightarrow A A^* K = K A^T \overline{A}$

$\Leftrightarrow (A A^* K)^* = (K A^T \overline{A})^*$

$\Leftrightarrow K^* (A^*)^* A^* = (\overline{A})^* (A^T)^* K^* \Leftrightarrow K A A^* = A^T \overline{A} K$

Pre and post multiply by K on both sides,

$\Leftrightarrow A A^* K = K A^T \overline{A} \Leftrightarrow A^*$ is con-k-normal.

(i) $\Leftrightarrow$ (v): A is con-k-normal $\Leftrightarrow A A^* K = K A^T \overline{A}$

$\Leftrightarrow (A A^* K)^{-1} = (K A^T \overline{A})^{-1}$

$\Leftrightarrow K^{-1} (A^*)^{-1} A^{-1} = (\overline{A})^{-1} (A^T)^{-1} K^{-1}$

$\Leftrightarrow K (A^{-1})^* A^{-1} = (\overline{A^{-1}})(A^{-1})^T K$

$\Leftrightarrow A^{-1}$ is con-k-normal, if A^{-1} exist.

(i) $\Leftrightarrow$ (vi): A is con-k-normal $\Leftrightarrow A A^* K = K A^T \overline{A}$

$\Leftrightarrow (A A^* K)^\dagger = (K A^T \overline{A})^\dagger$

$\Leftrightarrow K (A^\dagger)^* A^\dagger = (\overline{A^\dagger})(A^\dagger)^T K \Leftrightarrow A^\dagger$ is con-k-normal.

(i) $\Leftrightarrow$ (vii): A is k-normal $\Leftrightarrow A A^* K = K A^T \overline{A}$

$\Leftrightarrow \lambda^2 (A A^* K) = \lambda^2 (K A^T \overline{A})$

$\Leftrightarrow (\lambda A)(\lambda A^*) K = K (\lambda A^T)(\lambda \overline{A})$

$\Leftrightarrow (\lambda A)(\lambda A)^* K = K (\lambda A)^T \overline{(\lambda A)}$

$\Leftrightarrow (\lambda A)$ is con-k-normal.

Theorem 3.4:.If A and B are con-k-normal matrices. Then (A+B) and (A-B) are con-k-normal matrix.

Proof: Given A and B are con-k-normal. Then

$A A^* K = K A^T \overline{A}$ ------------- (1)

and $B B^* K = K B^T \overline{B}$ ------------- (2)

Adding equations (1) and (2), we get,

$A A^* K + B B^* K = K A^T \overline{A} + K B^T \overline{B}$

Pre and post multiply by $(A B^* + A^* B) K$ and $K(A^T \overline{B} + B^T \overline{A})$ we get,

$(AB^* + A^*B)K + (A A^* + B B^*)K = K(A^T \overline{A} + B^T \overline{B}) + K(A^T \overline{B} + B^T \overline{A})$

$\Rightarrow [A(A^* + B^*) + B(A^* + B^*)]K = K[(\overline{A} + \overline{B}) A^T + (\overline{A} + \overline{B}) B^T]$

$\Rightarrow (A+B)(A^* + B^*) K = K (A^T + B^T)(\overline{A} + \overline{B})$

$\Rightarrow (A+B)(A+B)^* K = K (A+B)^T \overline{(A+B)}$

Therefore $(A + B)$ is con-k-normal.

Similarly, we can prove $(A - B)$ is con-k-normal.

Theorem 3.5: Let $A \in \mathbb{C}_{n \times n}$,

(i). If A is con-k-normal, then (iA) is con-k-normal.
(ii). If A is con-k-normal, then (-iA) is con-k-normal.

Proof: (i) Given A is con-k-normal.

That is, $A A^* K = K A^T \overline{A} \Rightarrow i^2 (A A^* K) = i^2 (K A^T \overline{A})$

$\Rightarrow (iA)[-(iA)^*][\ ]K = K(iA)^T [-\overline{(iA)}]$

$\Rightarrow (iA)(iA)^* K = K(iA)^T \overline{(iA)}$

$\Rightarrow$ (iA) is con-k-normal.

Similarly, we can prove (-iA) is con-k-normal.

Theorem 3.6: Let $A \in \mathbb{C}_{n \times n}$ be con-k-normal, then $A\overline{A}$ and $\overline{A}A$ are k-normal.

Proof: Let A be con-k-normal, if $A A^* K = K A^T \overline{A}$.

We have, $(A\overline{A})(A\overline{A})^* K = (A\overline{A})(A^T A^*)K = A(\overline{A}A^T)A^* K$

$= A(K^2 \overline{A}A^T)A^* K$

$= A(A^* A)A^* K = (AA^*)^2 K$ …………(3)

and $K(A\overline{A})^* (A\overline{A}) = K(A^T A^*)(\overline{A}A) = (K A^T \overline{A})(A^* A)$

$= (AA^* K)(A^* A)$

$= (AA^*)(AA^*)K = (AA^*)^2 K$ …………….(4)

From (3) and (4), we have, $(A\overline{A})(A\overline{A})^* K = K(A\overline{A})^* (A\overline{A})$

Therefore, $A\overline{A}$ is k-normal.

Similarly, we can prove $\overline{A}A$ is k-normal.

Remark 3.7: The reverse implication $A\overline{A}$ and $\overline{A}A$ are k-normal, then A is con-k-normal is false.

Theorem 3.8[1]: Let $A, B \in \mathbb{C}_{n \times n}$ be con-k-normal matrices, then $A\overline{B} = B\overline{A} \Rightarrow A^T \overline{B} = BA^*$.

In words, if A is con-commutes with some matrix B, then A^T con-commutes with B as well.

Proof: Given A and B are con-k-normal matrices, then $A A^* K = K A^T \overline{A}$ and $B B^* K = K B^T \overline{B}$ .Since $A\overline{B} = B\overline{A}$ is trivially true for B=A. Let A be con-k-normal and let B be con-commute with A. $A\overline{B} = B\overline{A}$

For, $\hat{A} = \begin{bmatrix} 0 & A \\ \overline{A} & 0 \end{bmatrix}$ and $\hat{B} = \begin{bmatrix} 0 & B \\ \overline{B} & 0 \end{bmatrix}$. We have

$\hat{A}\hat{B} = A\overline{B} \oplus \overline{A}B$ and $\hat{B}\hat{A} = B\overline{A} \oplus \overline{B}A$.

$\Rightarrow \hat{A}$ and $\hat{B}$ commutes.

Since A is con-k-normal, then A^* is con-k-normal.

$\Rightarrow \hat{A}$ is con-k-normal, then $\Rightarrow \hat{A}^*$ is con-k-normal.

Therefore, $\hat{A}^* \hat{B} = \hat{B}\hat{A}^*$ which is equivalent to $A^T \overline{B} = BA^*$.

Con-k-unitary matrices

Definition 4.1: A matrix $A \in \mathbb{C}_{n \times n}$ is said to be con-k-unitary, if $A A^* K = K A^T \overline{A} = K$.

Example 4.2: Let $A = \begin{bmatrix} \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}$ is con-k-unitary for $k = (1, 2)$, where $K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Theorem 4.3: For $A \in \mathbb{C}_{n\times n}$ the following conditions are equivalent.

(i) A is con-k-unitary.

(ii) $\overline{A}$ is con-k- unitary.

(iii) A^T is con-k- unitary.

(iv) A^* is con-k- unitary.

(v) A^{-1} is con-k- unitary, if A^{-1} exist.

(vi) $A^{\dagger}$ is con-k- unitary.

(vii) λA is con-k- unitary, where λ is a real number.

Theorem 4.4: Let $A, B \in \mathbb{C}_{n\times n}$. If A and B are con-k-unitary matrices, then AB is con-k-normal.

Proof: Let A and B are con-k-unitary, then

$A A^* K = K A^T \overline{A} = K$ and $B B^* K = K B^T \overline{B} = K.$

To prove AB is con-k-normal.

Now, $(AB)(AB)^* K = AB(B^* A^*)K = A(BB^*)A^* K = AA^* K = K$ ……………..(5)

$K(AB)^T (\overline{AB}) = K(B^T A^T)(\overline{A}\overline{B}) = KB^T (A^T \overline{A})\overline{B} = KB^T \overline{B} = K$ …………(6)

From (5) and (6), we have

$(AB)(AB)^* K = K(AB)^T (\overline{AB}).$

Therefore AB is con-k-normal.

Corollary 4.5: Let $A, B \in \mathbb{C}_{n\times n}$. If A and B are con-k-unitary matrices, then AB is con-k-unitary.

Theorem 4.6: Let $A, B \in \mathbb{C}_{n\times n}$. If A and B are con-k-unitary matrices, then BA is con-unitary.

Proof: Let A and B are con-k-unitary, then

$A A^* K = K A^T \overline{A} = K$ …………(7)

and $B B^* K = K B^T \overline{B} = K$ …………(8)

From (7) and (8), we have

$A A^* K \; B B^* K = K A^T \overline{A} K B^T \overline{B} = K^2 = I$

$\Rightarrow \; KB B^* K = K A^T \overline{A} K = I$

$\Rightarrow \; B B^* = A^T \overline{A} = I$

$\Rightarrow \; B K^2 B^* = A^T K^2 \overline{A} = I$

$\Rightarrow \; B A A^* KKB^* = A^T KK B^T \overline{B}\overline{A} = I$

$\Rightarrow \; (BA)(BA)^* = (BA)^T (\overline{BA}) = I$

Therefore, BA is con-unitary.

Theorem 4.7: Let $A \in \mathbb{C}_{n\times n}$ and $A = SU, A^T = SV$, where U and V are con-k-unitary matrices and S is a symmetric matrix, then A is con-k-normal.

Proof: Let $A = SU$ and $A^T = SV$, where U and V are con-k-unitary.

Then,

$AA^* K = (SU)(SU)^* K = SUU^* \overline{S} K = SUU^* K^2 \overline{S} K = S\overline{S}K$ …………..(9)

$\overline{KA^*} A = KA^T \overline{A} = K(SV)(V^* \overline{S}) = SKVV^* \overline{S} = SV^* VK\overline{S} = SV^* V\overline{S}K = S\overline{S}K$ …………(10)

From (9) and (10), we have $AA^* K = \overline{KA^* A} = KA^T \overline{A}$.

Therefore, A is con-k-normal.

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