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International Journal of Current Research
Vol. 5, Issue, 02, pp.042-043, February, 2013

INTERNATIONAL JOURNAL OF CURRENT RESEARCH

ISSN: 0975-833X

RESEARCH ARTICLE

A Remark on Integral Operators Involving the Generalized M-Series

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ARTICLE INFO

Article History:
Received 11th November, 2012
Received in revised form
29th December, 2012
Accepted 16th January, 2013
Published online 14th February, 2013

Key words:
Fractional integral operators,
fractional differential operators,
generalized M-series.
Integral representation. 2000 Mathematics Subject Classification: 33E12, 26A33, 44A45.

ABSTRACT

The principal object of this paper is to prove three theorems based on integral representations of the generalized M-series which is introduced recently by Sharma and Jain[8].

1. INTRODUCTION

The Mittag-Leffler function has gained importance and popularity during the last one decade due mainly to its applications in the solution of fractional-order differential, integral and difference equations arising in certain problems of mathematical, physical, biological and engineering sciences. This function is introduced and studied by Mittag-Leffler[3] in terms of the power series

$$E_\alpha(x)=\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(\alpha n+1)},\ (\alpha>0) \tag{1.1}$$

A generalization of (1.1) in the following form

$$E_{\alpha,\beta}(x)=\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(\alpha n+\beta)},\quad (\alpha,\beta>0) \tag{1.2}$$

has been studied by several authors notably by Wiman[2]

In 2009, Sharma and Jain [8] introduced the M-series defined as

$${}_pM_q^{\alpha,\beta}(a_1,\ldots,a_p,b_1,\ldots,b_q;x)={}_pM_q^{\alpha,\beta}(x)=\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n}{(b_1)_n\ldots(b_q)_n}\frac{x^n}{\Gamma(\alpha n+\beta)} \tag{1.3}$$

where $\alpha,\beta\in C, R(\alpha)>0$ and $(a_i)_n(i=1,2,\ldots,p)$ and $(b_j)_n\ (j=1,2,\ldots,q)$ are the Pochhammer symbols. Further details of this series are given by [8]. The Riemann-Liouville operators of fractional calculus are defined in the books by Miller and Ross[7], Oldham and Spanier[6], Podlubny[5], Kilbas, Srivastava and Trujillo[1] as

$${}_aD_t^{-\nu}f(t)=\frac{1}{\Gamma(\nu)}\int_a^t(t-x)^{\nu-1}f(x)dx,\mathrm{Re}(\nu)>0,t>a \tag{1.4}$$

By virtue of above definition, it is not difficult to show that

$${}_aD_t^{-\nu}(t-a)^{\rho-1}=\frac{\Gamma(\rho)}{\Gamma(\rho+\nu)}(t-a)^{\nu+\rho-1},\mathrm{Re}(\nu)>0,\mathrm{Re}(\rho)>0;t>a. \tag{1.5}$$

Also from Podlubny[5], we have

$${}_aD_t^{\nu}(t-a)^{\rho-1}=\frac{\Gamma(\rho)}{\Gamma(\rho-\nu)}(t-a)^{-\nu+\rho-1},\mathrm{Re}(\nu)>0,\mathrm{Re}(\rho)>0;t>a. \tag{1.6}$$

If we take $\rho=1$ in (1.6), it reduces to

$${}_aD_t^{\nu}1=\frac{1}{\Gamma(1-\nu)}(t-a)^{-\nu},t>a;\nu\neq 1,2,\ldots \tag{1.7}$$

which is a remarkable result in the theory of fractional calculus and indicates that the fractional derivative of a constant in the Riemann-Liouville sense is not zero.

For $a=0$, (1.4) reduces to

$${}_0D_t^{-\nu}f(t)=\frac{1}{\Gamma(\nu)}\int_0^t(t-x)^{\nu-1}f(x)dx,\mathrm{Re}(\nu)>0. \tag{1.8}$$

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2. THE INTEGRAL REPRESENTATION OF THE GENERALIZED M-SERIES

In this section, we derive formulae based on integral representations of the generalized M-series. The results are presented in the form of the theorems given below:

Theorem 2.1 If

$\alpha, \beta \in C, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, \beta > \alpha > 0$ then

$$ {}_pM_q^{\alpha,\beta}(z) = k\, z^{\alpha-\beta} \int_0^\infty e^{\left(-\frac{t^k}{z^k}\right)} t^{\beta-\alpha-1} \sum_{n=0}^\infty \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{t^n}{\Gamma(\alpha n+\beta)\Gamma\left(\frac{\beta-\alpha+n}{k}\right)} dt \tag{2.1} $$

where ${}_pM_q^{\alpha,\beta}(z)$ is the generalized M-sersies given by (1.3).

Proof: On taking the term

$$ \int_0^\infty e^{\left(-\frac{t^k}{z^k}\right)} t^{\beta-\alpha-1} \sum_{n=0}^\infty \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{t^n}{\Gamma(\alpha n+\beta)\Gamma\left(\frac{\beta-\alpha+n}{k}\right)} dt \tag{2.2} $$

Now interchanging the order of summation and integrssation which is permissible under the stated conditions and putting $\frac{t^k}{z^k} = u$, we get

$$ = \sum_{n=0}^\infty \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{z^{\beta-\alpha+n}}{\Gamma(\alpha n+\beta)\Gamma\left(\frac{\beta-\alpha+n}{k}\right)} \frac{1}{k} \int_0^\infty e^{-u} u^{\frac{\beta-\alpha+n}{k}-1} du \tag{2.3} $$

On applying(1.3), it redsuces to

$$ = \frac{z^{\beta-\alpha}}{k} {}_pM_q^{\alpha,\beta}(z) \tag{2.4} $$

Therefore

$$ {}_pM_q^{\alpha,\beta}(z) = k\, z^{\alpha-\beta} \int_0^\infty e^{\left(-\frac{t^k}{z^k}\right)} t^{\beta-\alpha-1} \sum_{n=0}^\infty \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{t^n}{\Gamma(\alpha n+\beta)\Gamma\left(\frac{\beta-\alpha+n}{k}\right)} dt \tag{2.5} $$

This completes the proof.

Theorem 2.2 If

$\alpha, \beta \in C, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, \beta > \alpha > 0$ then

$$ {}_pM_q^{\alpha,\beta}(z) = \frac{1}{\alpha\Gamma(\beta-\alpha)} \int_0^1 (1-t^{\frac{1}{\alpha}})^{\beta-\alpha-1} {}_pM_q^{\alpha,\alpha}(tz) dt \tag{2.6} $$

where ${}_pM_q^{\alpha,\beta}(z)$ is the generalized M-series given by (1.3).

Proof: Consider

$$ \int_0^1 (1-t^{\frac{1}{\alpha}})^{\beta-\alpha-1} {}_pM_q^{\alpha,\alpha}(tz) dt\,. \tag{2.7} $$

On interchanging the order of summation and integration which is permissible under the stated conditions, using (1.3) and substituting $t^{\frac{1}{\alpha}} = u$, we obtain

$$ = \alpha \sum_{n=0}^\infty \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{z^n}{\Gamma(\alpha n+\alpha)} \int_0^1 u^{\alpha n+\alpha-1} (1-u)^{\beta-\alpha-1} du \tag{2.8} $$

On evaluating the inner integral with the help of Beta function, we arrive at

$$ = \alpha\Gamma(\beta-\alpha) \sum_{n=0}^\infty \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{z^n}{\Gamma(\alpha n+\beta)} \tag{2.9} $$

Again using(1.3), it reduces to

$$ = \alpha\Gamma(\beta-\alpha)\, {}_pM_q^{\alpha,\beta}(z) \tag{2.10} $$

Therefore

$$ {}_pM_q^{\alpha,\beta}(z) = \frac{1}{\alpha\Gamma(\beta-\alpha)} \int_0^1 (1-t^{\frac{1}{\alpha}})^{\beta-\alpha-1} {}_pM_q^{\alpha,\alpha}(tz) dt \tag{2.11} $$

This proves theorem(2.2).

Theorem 2.3 If

$\alpha, \beta \in C, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, \beta > \alpha > 0$ then

$$ {}_pM_q^{\alpha,\beta}(z) = \frac{1}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-\alpha-1} {}_pM_q^{\alpha,\beta-\alpha}[z(1-t)^\alpha] dt \tag{2.12} $$

where ${}_pM_q^{\alpha,\beta}(z)$ is the generalized M-series given by (1.3).

Proof: Same as above theorem.

3. Special Cases

Theorems (2.1) to (2.3) leads to the integral representation of M-series[9], generalized Mittag-Leffler functions[2], Mittag-Leffler function[3] and exponential function[4] after implementing the necessary changes in the values of p, q, α and β.

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