AN ORTHOGONAL GENERALIZED HIGHER REVERSE LEFT (RESP. RIGHT) CENTRALIZER ON SEMIPRIME RINGS

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ABSTRACT

The main object of this paper is to prove that: Let R be a 2-torsion free semiprime ring, T=(T_i)_{i \in \mathbb{N}} and H=(H_i)_{i \in \mathbb{N}} be two generalized higher reverse left (resp. right) centralizers associated with the higher reverse left (resp. right) centralizers t=(t_i)_{i \in \mathbb{N}} and h=(h_i)_{i \in \mathbb{N}} resp. of R , where T_n and H_n are commuting. Then T_n and H_n are orthogonal if and only if T_n(x) H_n(y) = t(x) H_n(y) = 0, for all x , y ∈ R and n ∈ N.

INTRODUCTION

A ring R is called semiprime if \(xRx = (0)\) implies \(x = 0\), such that \(x \in R (3)\). Let R be a ring then R is called 2-torsion free if \(2x = 0\) implies \(x = 0\), for all \(x \in R (3)\). Zalar (5) present the concepts of centralizer and Jordan centralizer of a ring R as follows:

A left (resp. right) centralizer of a ring R is an additive mapping \(t: R \rightarrow R\) which satisfies the following equation \(t(xy) = t(x)y + x t(y)\), for all \(x, y \in R\). T is called a centralizer of R if it is both a left and a right centralizer. A left (resp. right) Jordan centralizer of a ring R is an additive mapping \(t: R \rightarrow R\) which satisfies the following equation \(t(x^2) = x t(x)\), for all \(x \in R\). T is called a Jordan centralizer of R if it is both a left and a right Jordan centralizer.

Jarullah and Salih (4) introduced the concepts of a generalized higher reverse left (resp. right) centralizer and a Jordan generalized higher reverse left (resp. right) centralizer on rings as follows:

Let \(T = (T_i)_{i \in \mathbb{N}}\) be a family of additive mappings of a ring R into itself. Then T is called a generalized higher reverse left (resp. right) centralizer associated with the higher reverse left (resp. right) centralizer \(t = (t_i)_{i \in \mathbb{N}}\) of R if for all \(x, y \in R\) and \(n \in \mathbb{N}\)

\[T_n(x y) = \sum_{i=1}^{n} T_i(y) t_{i-1}(x)\]

(esp. \(T_n(x y) = \sum_{i=1}^{n} t_{i-1}(y) T_i(x)\)).

Let \(T = (T_i)_{i \in \mathbb{N}}\) be a family of additive mappings of a ring R into itself. Then T is called a Jordan generalized higher reverse left (resp. right) centralizer associated with the Jordan higher reverse left (resp. right) centralizer \(t = (t_i)_{i \in \mathbb{N}}\) of R , if the following equation holds, for all \(x \in R\) and \(n \in \mathbb{N}\):

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In this paper, we define and study the concept of orthogonal generalized higher reverse left (resp. right) centralizers of semiprime rings and we prove some of lemmas and theorems about orthogonally one of these Theorems is: Let $R$ be a 2-torsion free semiprime ring $T=(T_t)_{t \in \mathbb{N}}$ and $H=(H_t)_{t \in \mathbb{N}}$ be two generalized higher reverse left (resp. right) centralizers of $R$, Suppose that $T^n_t = H^n_t$, for all $t \in \mathbb{N}$. Then $T^n_t + H_t$ and $T^n_t - H_t$ are orthogonal. In our work we need the following Lemmas:

**Lemma (1.1): (2)**

Let $R$ be a 2-torsion free semiprime ring and $x, y$ be elements of $R$, then the following conditions are equivalent:

1. $xry = 0$, for all $r \in R$ (i)
2. $xry = 0$, for all $r \in R$ (ii)
3. $xry = 0$, for all $r \in R$ (iii)

If one of these conditions is fulfilled, then $xy = yx = 0$.

**Lemma (1.2): (1)**

Let $R$ be a 2-torsion free semiprime ring and $x, y$ be elements of $R$ if $xry + yrx = 0$, then $xy = yx = 0$.

**Orthogonal Generalized Higher Reverse Left (resp. Right) Centralizers on Semiprime Rings:**

**Semiprime Rings:** In this section we will introduce and study the concept of orthogonal generalized higher reverse left (resp. right) centralizers on semiprime rings.

**Definition (2.1):**

Two generalized higher reverse left (resp. right) centralizers $T=(T_t)_{t \in \mathbb{N}}$ and $H=(H_t)_{t \in \mathbb{N}}$ of a ring $R$ are called orthogonal if $T_n(x)R H_n(y) = (0) = H_n(y)R T_n(x)$, for all $x, y \in R$ and $n \in \mathbb{N}$. Where $T_n(x)R H_n(y) = \sum_{i=1}^{n} T_i(x)z H_i(y)$, for all $z \in R$.

**Lemma (2.2):**

Let $R$ be a semiprime ring, suppose that $T=(T_t)_{t \in \mathbb{N}}$ and $H=(H_t)_{t \in \mathbb{N}}$ be two generalized higher reverse left (resp. right) centralizers of $R$, satisfy $T_n(x)R H_n(x) = (0)$, for all $x \in R$ and $n \in \mathbb{N}$. Then $T_n(x)R H_n(y) = (0)$, for all $x, y \in R$ and $n \in \mathbb{N}$.

**Proof:**

Suppose that $T_n(x)R H_n(x) = (0)$, for all $x \in R$ and $n \in \mathbb{N}$ That is $T_n(x)R H_n(x) = \sum_{i=1}^{n} T_i(x)z H_i(x) = 0$, for all $x, z \in R$ ...(1)

Replace $x$ by $x + y$ in (1), we have that $\sum_{i=1}^{n} T_i(x+y)z H_i(x+y) = 0 \sum_{i=1}^{n} T_i(x)z H_i(x) + T_i(x)z H_i(y) + T_i(y)z H_i(x) + T_i(y)z H_i(y) = 0$ Therefore, by our assumption and Lemma (1.1), we get

$\sum_{i=1}^{n} T_i(x)z H_i(x) = 0$, for all $x, y, z \in R$

Thus, $T_n(x)R H_n(y) = (0)$, for all $x, y \in R$ and $n \in \mathbb{N}$.

**Lemma (2.3):**

Let $R$ be a 2-torsion free semiprime ring, $T=(T_t)_{t \in \mathbb{N}}$ and $H=(H_t)_{t \in \mathbb{N}}$ be two generalized higher reverse left (resp. right) centralizers of $R$. Then $T_n$ and $H_n$ are orthogonal if and only if $T_n(x)H_n(y) + H_n(x)T_n(y) = 0$, for all $x, y \in R$ and $n \in \mathbb{N}$.

**Proof:** Suppose that $T_n$ and $H_n$ are orthogonal. Then $T_n(x)H_n(y) + H_n(x)T_n(y) = 0$, for all $x, y \in R$ and $n \in \mathbb{N}$ Since $T_n$ and $H_n$ are orthogonal, we have that $\sum_{i=1}^{n} T_i(x)z H_i(y) = 0 = \sum_{i=1}^{n} H_i(y)z T_i(x)$, for all $x, y, z \in R$ Therefore, by Lemma (1.1), we get the require result.

Conversely, it's clear by using Lemma (1.2)
Theorem (2.4): Let \( R \) be a 2-torsion free semiprime ring, \( T=(T_i)_{i \in \mathbb{N}} \) and \( H=(H_i)_{i \in \mathbb{N}} \) are orthogonal generalized higher reverse left (resp.right) centralizers associated with the higher reverse left (resp.right) centralizers \( t=(t_i)_{i \in \mathbb{N}} \) and \( h=(h_i)_{i \in \mathbb{N}} \) resp. of \( R \), where \( T_n \) and \( H_n \) are commuting. Then the following relations are holds, for all \( x, y \in R \) and \( n \in \mathbb{N} \): (i) \( T_n(x) H_n(y) = T_n(x) T_n(y) = 0 \) Hence \( T_n(x) H_n(y) + H_n(x) T_n(y) = 0 \) (iii) \( t_n \) and \( H_n \) are orthogonal and \( t_n(x) H_n(y) = H_n(x) t_n(y) = 0 \) (iii) \( h_n \) and \( T_n \) are orthogonal and \( h_n(x) T_n(y) = T_n(x) h_n(y) = 0 \) (iv) \( t_n \) and \( H_n \) are orthogonal higher reverse left (resp.right) centralizers

**Proof:**

(i) Suppose that \( T_n \) and \( H_n \) are orthogonal \( \sum_{i=1}^{n} T_i(x) z H_i(y) = 0 = \sum_{i=1}^{n} H_i(y) z T_i(x) \), for all \( x, y, z \in R \) By Lemma (1.1), we have that \( \sum_{i=1}^{n} T_i(x) H_i(y) = \sum_{i=1}^{n} H_i(y) T_i(x) = 0 \), for all \( x, y \in R \) Then, we get \( \sum_{i=1}^{n} T_i(x) H_i(y) + H_i(x) T_i(y) = 0 \), for all \( x, y \in R \), for all \( n \in \mathbb{N} \).

(ii) Since \( T_n \) and \( H_n \) are orthogonal. From (2), we have that \( \sum_{i=1}^{n} H_i(y) t_{i+1}(z) = 0 \) By Lemma (1.1), we have that \( \sum_{i=1}^{n} H_i(y) t_{i+1}(z) = 0 \).

Right multiply by \( t_i(x) \), we have that

\[
\sum_{i=1}^{n} H_i(y) t_{i+1}(z) t_i(x) = 0, \text{ for all } x, y, z \in R \text{ ... (1) Since } H_n \text{ is a commuting, we have that } \sum_{i=1}^{n} t_i(x) t_{i+1}(z) H_i(y) = 0, \text{ for all } x, y, z \in R \text{ ... (2) By (1) and (2), we get } t_n \text{ and } H_n \text{ are orthogonal . From (2), we have that } \sum_{i=1}^{n} t_i(x) t_{i+1}(z) H_i(y) = 0, \text{ for all } x, y, z \in R \text{ By Lemma (1.1), we have that } \sum_{i=1}^{n} t_i(x) H_i(y) = 0, \text{ for all } x, y, z \in R \text{ ... (3) By the same method as (ii). (iv) Since } T_n \text{ and } H_n \text{ are orthogonal } \sum_{i=1}^{n} t_i(x) H_i(y) = 0, \text{ for all } x, y, z \in R \text{, and } n \in \mathbb{N} \text{ Replace } y \text{ by } yz \text{, we have that } \sum_{i=1}^{n} t_i(x) H_i(y) = 0, \text{ for all } x, y, z \in R \text{. By Lemma (1.1), we get the require result}.

Theorem (2.5)

Let \( R \) be a 2-torsion free semiprime ring, \( T=(T_i)_{i \in \mathbb{N}} \) and \( H=(H_i)_{i \in \mathbb{N}} \) are orthogonal generalized higher reverse left (resp.right) centralizers associated with the higher reverse left (resp.right) centralizers \( t=(t_i)_{i \in \mathbb{N}} \) and \( h=(h_i)_{i \in \mathbb{N}} \) resp. of \( R \). Then the following relations are holds, for all \( n \in \mathbb{N} \): (i) \( t_n \) \( H_n \) and \( t_n H_n = T_n h_n = 0 \) (ii) \( h_n T_n = T_n h_n = 0 \) (iii) \( T_n H_n = h_n T_n = 0 \)

**Proof:** (i) Since \( T_n \) and \( H_n \) are orthogonal By Theorem (2.4)(ii), we have that \( t_n(x) H_n(y) = 0 \), for all \( x, y \in R \) and \( n \in \mathbb{N} \)

\[
\sum_{i=1}^{n} t_i(x) H_i(y) = 0, \text{ for all } x, y \in R \sum_{i=1}^{n} t_i(t_i(x) H_i(y)) = 0 \sum_{i=1}^{n} t_i(H_i(y)) t_{i+1}(t_i(x)) = 0 \text{ Right multiply by } t_i(H_i(y)) \text{, we have that } \sum_{i=1}^{n} t_i(H_i(y)) t_{i+1}(t_i(x)) t_i(H_i(y)) = 0, \text{ for all } x, y \in R \text{ Since } R \text{ is a semiprime ring, we have that } \sum_{i=1}^{n} t_i(H_i(y)) = 0 \]

for all \( y \in R \Rightarrow t_n H_n = 0 \), for all \( n \in \mathbb{N} \) ... (1) Also, by Theorem (2.4)(ii), we have that \( H_n(x) t_n(y) = 0 \), for all \( x, y \in R \) and \( n \in \mathbb{N} \)
\[
\sum_{i=1}^{n} H_i(x) t_i(y) = 0 \quad \sum_{i=1}^{n} H_i(t_i(y)) h_{i.1}(H_i(x)) = 0 \quad \text{Right multiply by } H_i(t_i(y)) \quad \text{we have that}
\]
\[
\sum_{i=1}^{n} H_i(t_i(y)) h_{i.1}(H_i(x)) = 0 \quad \text{Since } R \text{ is a semiprime ring, we have that}
\]
\[
\sum_{i=1}^{n} H_i(t_i(y)) = 0 \quad \text{for all } y \in R \Rightarrow H_n t_n = 0, \quad \text{for all } n \in N \ldots (2)
\]
From (1) and (2), we get \( t_n H_n = H_n t_n = 0 \), for all \( n \in N \). (ii) By the same method as (i) (iii) Since \( T_n \) and \( H_n \) are orthogonal By Theorem (2.4)(i), we have that \( T_n(x) H_n(y) = 0 \), for all \( x, y \in R \) and \( n \in N \)
\[
\sum_{i=1}^{n} T_i(x) H_i(y) = 0 \quad \sum_{i=1}^{n} T_i(H_i(y)) t_i(x) = 0 \quad \text{Right multiply by } T_i(H_i(y)) \quad \text{we have that}
\]
\[
\sum_{i=1}^{n} T_i(H_i(y)) t_i(x) (T_i(x)) = 0 \quad \text{Since } R \text{ is a semiprime ring, we have that}
\]
\[
\sum_{i=1}^{n} T_i(H_i(y)) = 0 \quad \text{for all } y \in R \Rightarrow T_n H_n = 0, \quad \text{for all } n \in N \ldots (1)
\]
Also, By Theorem (2.4)(i), we have that \( H_n(x) T_n(y) = 0 \), for all \( x, y \in R \) and \( n \in N \)
\[
\sum_{i=1}^{n} H_i(x) T_i(y) = 0 \quad \sum_{i=1}^{n} H_i(H_i(x) T_i(y)) = 0
\]
\[
\sum_{i=1}^{n} H_i(T_i(y)) h_{i.1}(H_i(x)) = 0 \quad \text{Right multiply by } H_i(T_i(y)) \quad \text{we have that}
\]
\[
\sum_{i=1}^{n} H_i(T_i(y)) h_{i.1}(H_i(x)) H_i(T_i(y)) = 0 \quad \text{Since } R \text{ is a semiprime ring, we have that}
\]
\[
\sum_{i=1}^{n} H_i(T_i(y)) = 0 \quad \text{for all } y \in R \Rightarrow H_n T_n = 0, \quad \text{for all } n \in N.
\]

**Theorem (2.6):**

Let \( R \) be a 2-torsion free semiprime ring, \( T=(T_i)_{i \in N} \) and \( H=(H_i)_{i \in N} \) be two generalized higher reverse left (resp. right) centralizers associated with the higher reverse left (resp. right) centralizers \( t=(t_i)_{i \in N} \) and \( h=(h_i)_{i \in N} \) resp. of \( R \), where \( T_n \) and \( H_n \) are commuting. Then \( T_n \) and \( H_n \) are orthogonal if and only if the following relations are holds, for all \( x, y \in R \) and \( n \in N \):
(i) \( T_n(x) H_n(y) + H_n(x) T_n(y) = 0 \) (ii) \( t_n(x) H_n(y) = h_n(x) T_n(y) = 0 \)

**Proof:**

Suppose that \( T_n \) and \( H_n \) are orthogonal T.P. (i) \( T_n(x) H_n(y) + H_n(x) T_n(y) = 0 \)

(ii) \( t_n(x) H_n(y) = h_n(x) T_n(y) = 0 \), for all \( x, y \in R \) and \( n \in N \)

Since \( T_n \) and \( H_n \) are orthogonal

By Lemma (2.3), we get (i).

Now, Since \( T_n \) and \( H_n \) are orthogonal.

By Theorem (2.4)(i), we have that \( T_n(x) H_n(y) = 0 \)

Replace \( H_i(y) \) by \( x \), we have that
\[
\sum_{i=1}^{n} T_i(x) x = 0
\]
\[
\sum_{i=1}^{n} t_i (T_i(x) x) = 0
\]
\[
\sum_{i=1}^{n} t_i(x) t_{i.1}(T_i(x)) = 0
\]
Left multiply by \( H_i(y) \) and since \( H_i \) is a commuting, we have that
\[
\sum_{i=1}^{n} t_i(x) H_i(y) t_{i.1}(T_i(x)) = 0
\]
Right multiply by \( t_i(x) H_i(y) \), we have that
\[
\sum_{i=1}^{n} t_i(x) H_i(y) t_{i.1}(T_i(x)) t_i(x) H_i(y) = 0
\]
From (1) and (2), we get the required result.

Since R is a semiprime ring, we have that $\sum_{i=1}^{n} t_i(x) H_i(y) = 0 \Rightarrow t_0(x) H_n(y) = 0$, for all $x, y \in R$ and $n \in N$ ...(1)

Also, by Theorem (2.4)(i), we have that $H_n(x) T_n(y) = 0$, for all $x, y \in R$ and $n \in N$.

$$\sum_{i=1}^{n} H_i(x) T_i(y) = 0 \Rightarrow t_0(x) H_n(y) = 0$$

Replace $T_i(y)$ by $x$, we have that

$$\sum_{i=1}^{n} H_i(x) x = 0$$

$$\sum_{i=1}^{n} h_i(H_i(x)) = 0$$

Left multiply by $T_i(y)$ and since $T_n$ is a commuting, we have that

$$\sum_{i=1}^{n} h_i(x) T_i(y) = 0$$

Right multiply by $h_i(x) T_i(y)$, we have that

$$\sum_{i=1}^{n} h_i(x) T_i(y) h_i(H_i(x)) = 0$$

Since $R$ is a semiprime ring, we have that

$$\sum_{i=1}^{n} h_i(x) T_i(y) = 0 \Rightarrow h_i(x) T_n(y) = 0$$

For all $x, y \in R$ and $n \in N$ ...(2)

From (1) and (2), we get (ii).

Conversely, Suppose that the relations are hold, for all $x, y \in R$ and $n \in N$:

(i) $T_n(x) H_n(y) + H_n(x) T_n(y) = 0$

(ii) $t_0(x) H_n(y) = h_0(x) T_n(y) = 0$ T.P. $T_n$ and $H_n$ are orthogonal

From (i), we have that $T_n(x) H_n(y) + H_n(x) T_n(y) = 0$, for all $x, y \in R$ and $n \in N$.

**Theorem (2.7):** Let $R$ be a 2-torsion free semiprime ring, $T=(T_i)_{i \in N}$ and $H=(H_i)_{i \in N}$ be two generalized higher reverse left (resp. right) centralizers associated with the higher reverse left (resp. right) centralizers $t=(t_i)_{i \in N}$ and $h=(h_i)_{i \in N}$ resp. of $R$, where $T_n$ and $H_n$ are commuting. Then $T_n$ and $H_n$ are orthogonal if and only if $T_n(x) H_n(y) = t_0(x) H_n(y) = 0$, for all $x, y \in R$ and $n \in N$.

**Proof:** Suppose that $T_n$ and $H_n$ are orthogonal. T.P. $T_n(x) H_n(y) = t_0(x) H_n(y) = 0$, for all $x, y \in R$ and $n \in N$. Since $T_n$ and $H_n$ are orthogonal, By Theorem (2.4)(i), we have that $T_n(x) H_n(y) = 0$, for all $x, y \in R$ and $n \in N$ ...(1) $\sum_{i=1}^{n} T_i(x) H_i(y) = 0$, for all $x, y \in R$. Replace $H_i(y)$ by $x$, we have that $\sum_{i=1}^{n} t_i(T_i(x)) x = 0$ $\sum_{i=1}^{n} t_i(x) t_{i+1}(T_i(x)) = 0$ Left multiply by $H_i(y)$ and since $H_i$ is a commuting, we have that $\sum_{i=1}^{n} t_i(x) H_i(y) t_{i+1}(T_i(x)) = 0$ Right multiply by $t_i(x) H_i(y)$, we have that

$$\sum_{i=1}^{n} t_i(x) H_i(y) t_{i+1}(T_i(x)) t_i(x) H_i(y) = 0$$

Since $R$ is a semiprime ring, we have that

$$\sum_{i=1}^{n} t_i(x) H_i(y) = 0 \Rightarrow t_0(x) H_n(y) = 0$$

For all $x, y \in R$ and $n \in N$ ...(2)

From (1) and (2), we get the required result.
Conversely, suppose that $T_n(x) H_n(y) = T_n(x) H_n(y) = 0$, for all $x, y \in R$ and $n \in N$. T.P. $T_n$ and $H_n$ are orthogonal. By assumption, we have that $T_n(x) H_n(y) = 0$ for all $x, y \in R$ and $n \in N$. Hence $T_n$ and $H_n$ are orthogonal.

**Theorem (2.8)**

Let $R$ be a 2-torsion free semiprime ring, $T=(T_i)_{i \in N}$ and $H=(H_i)_{i \in N}$ be two generalized higher reverse left (resp. right) centralizers associated with the higher reverse left (resp. right) centralizers $t=(t_i)_{i \in N}$ and $h=(h_i)_{i \in N}$ of $R$, where $T_n$ and $H_n$ are commuting. Then $T_n$ and $H_n$ are orthogonal if and only if $T_n(x) H_n(y) = 0$, and $t_i H_n = t_n h_i = 0$, for all $x, y \in R$ and $n \in N$.

**Proof:**

Suppose that $T_n$ and $H_n$ are orthogonal. T.P. $T_n(x) H_n(y) = 0$ and $t_i H_n = t_n h_i = 0$, for all $x, y \in R$ and $n \in N$. Since $T_n$ and $H_n$ are orthogonal by Theorem (2.4)(i), we have that $T_n(x) H_n(y) = 0$, for all $x, y \in R$ and $n \in N$. By Theorem (2.5)(i), we have that $t_i H_n = 0$, for all $n \in N$. By Theorem (2.4)(iii), we have that $T_n(x) h_i H_n(y) = 0$, for all $x, y \in R$ and $n \in N$.

Let $T_n(x) H_n(y) = 0$. By assumption, we have that $t_i H_n = 0$, for all $x \in R$. For all $n \in N$.

From (1) and (2), we get the required result. Conversely, suppose that $T_n(x) H_n(y) = 0$ and $t_i H_n = t_n h_i = 0$, for all $x, y \in R$ and $n \in N$. T.P. $T_n$ and $H_n$ are orthogonal. By assumption, we have that $T_n(x) H_n(y) = 0$ for all $x, y \in R$ and $n \in N$. By Theorem (2.4)(iii), we have that $T_n(x) h_i H_n(y) = 0$, for all $x, y \in R$ and $n \in N$. From (4) and (5), we get $T_n$ and $H_n$ are orthogonal.

**Theorem (2.9):** Let $R$ be a 2-torsion free semiprime ring, $T=(T_i)_{i \in N}$ and $H=(H_i)_{i \in N}$ be two generalized higher reverse left (resp. right) centralizers of $R$. Suppose that $T_n = H_n$, for all $n \in N$. Then $T_n + H_n$ and $T_n - H_n$ are orthogonal.

**Proof:**

\[
( T_n + H_n)(T_n - H_n) + (T_n - H_n)(T_n + H_n) \]

\[
= \sum_{i=1}^{n} T_i(x) - T_i(x) H_i(x) + H_i(x) T_i(x) - H_i(x) T_i(x) + T_i(x) H_i(x) - H_i(x) T_i(x) - H_i(x) T_i(x) = 0
\]

Therefore, \( ( T_n + H_n)(T_n - H_n) + (T_n - H_n)(T_n + H_n) \)(x) = 0

By Lemma (2.3), we get the required result.

**REFERENCES**


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