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RESEARCH ARTICLE

WAVELETS IN WEIGHTED SOBOLEV SPACE

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ABSTRACT

Wavelet transform is studied on the weighted Sobolev space B_k^{ω}. Boundedness results in this Sobolev space is obtained. Wavelet tra
nsform with compactly supported wavelet is studied. Asymptotic properties of the wavelet transform will also be discussed.

INTRODUCTION

Let $\psi \in L^2(\mathbf{R})$ be the analyzing wavelet and $f \in L^2(\mathbf{R})$ be any function. We define the translation operator T_b by

$$T_b\psi(x) = \psi(x-b), \quad b \in \mathbf{R} \tag{1.1}$$

and the dilation operator D_a by

$$D_a\psi(x) = |a|^{-1/2}\,\psi\left(\frac{x}{a}\right),\ a \in \mathbf{R}_0 = \mathbf{R}\backslash\{0\}. \tag{1.2}$$

and a unitary transformation $U(b,a) : L^2(\mathbf{R}, dt) \rightarrow L^2(\mathbf{R}, dt)$ by

$$U(b,a)\,\psi(x) = (T_b D_a \psi)(x) = |a|^{-1/2}\,\psi\left(\frac{x-b}{a}\right), (b,\ a) \in \mathbf{R} \text{ x } \mathbf{R}_0.$$

The actions of the Fourier transform

$$(Ff)(\xi) = \hat{f}(\xi) = \int_R f(x)e^{-i\xi x}dx, \quad \xi \in \mathbf{R} \tag{1.3}$$

on the operators T_b and D_a are given by

$$FT_b = e^{-b(.)}F, \tag{1.4}$$

$$FD_a = D_{1/a}F. \tag{1.5}$$

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Definition1.1. A function $\psi \in L^2(\mathbf{R}, dt)$ is admissible only if ψ is not identical to zero and

$$\int_{R_0}\int_{R} \left| < U(b,a)\psi, \psi >_0 \right|^2 \frac{dadb}{a^2} < \infty . \tag{1.6}$$

Lemma1.2. $\psi \in L^2(\mathbf{R}, dt)\backslash\{0\}$ is admissible if and only if the integral $\int_{R_0} \frac{|\hat{\psi}(\xi)|^2}{|\xi|} d\xi$ exists.

Prof. See [4, p. 877].

Lemma1.3. Let ψ be admissible and $f \in L^2(\mathbf{R}, dt)$

Let

$$C_\psi = \int_{R_0} \frac{\left|\hat{\psi}(\xi)\right|^2}{|\xi|} d\xi.$$

The integral

$$(L_\psi f)(b,a) = \tilde{f}(b,a) = \frac{1}{\sqrt{C\psi}} < f, U(b,a)\psi >_0 \tag{1.7}$$

$$= \frac{1}{\sqrt{C\psi}} \frac{1}{\sqrt{|a|}} \int_{R} \overline{\psi}\left(\frac{t-b}{a}\right) f(t) dt \tag{1.8}$$

defines an element of $L^2 \left(\mathbf{R} x \mathbf{R}_0, \frac{dbda}{a^2}\right)$.

Moreover,

$L_\psi : L^2(\mathbf{R}, dt) \to L^2 \left(\mathbf{R} x \mathbf{R}_0, \frac{dbda}{a^2}\right)$ is an isometry.

Proof. See [4, p, 877).

From (1.5), the Fourier transform of L_ψ with respect to its translation argument is given by

$$(L_\psi f)(.,a)^\wedge(\xi) = \sqrt{\frac{1}{C_\psi}} |a|^{\frac{1}{2}} \hat{\psi}(-a\xi) \hat{f}(\xi). \tag{1.9}$$

Definition1.4. The operator $L_\psi : L^2(\mathbf{R}, dt) \to L^2 \left(\mathbf{R} x \mathbf{R}_0, \frac{dbda}{a^2}\right)$ is called wavelet transform with respect to analyzing wavelet ψ.

In this paper, we extend the wavelet transform, which we defined on $L^2(\mathbf{R}, dt)$, to weighted Sobolev space $\mathbf{B}_k^\omega$ and boundedness properties will be investigated. Asymptotic properties for small dilation parameter will also be studied.

The Weighted Sobolev Space B_k^ω.

In this section we recall definitions and properties of certain function and distribution spaces introduced by BjÖrck [1]. Let M be the set of continuous and real valued functions ω on $\mathbf{R}^n$ satisfying the following conditions:

(1) $0 = \omega(0) \le \omega(\xi + n) \le \omega(\xi) + \omega(\eta);\ \xi, \eta \in \mathbf{R}^n$ (2.1)

(2) $\int_0^\infty \frac{\omega(\xi)}{(1+|\xi|)^{n+1}} < \infty$, (2.2)

(3) $\omega(\xi) \ge a + b\ \log\ (1+|\xi|),\ \ \xi \in \mathbf{R}^n$ (2.3)

for some real number a and position real number b. We denote by M_c the set of all $\omega \in M$ satisfying condition $\omega(\xi) = \Omega(|\xi|)$ with a concave function Ω on $[0,\infty)$ We suppose $\omega \in M_c$ from now on.

Let $\omega \in M_c$. We denote by S_ω the set of all functions $\phi \in L^1(\mathbf{R}^n)$ with the property that ϕ and $\hat{\phi} \in C^\infty$ and for each multi index α and each non-negative number λ we have

$$p_{\alpha,\lambda}(\phi) = \sup_{x \in R^n} e^{\lambda\omega(x)} |D^\alpha \phi(x)| < \infty; \tag{2.4}$$

$$\pi_{\alpha,\lambda}(\phi) = \sup_{\xi \in R^n} e^{\lambda\omega(\xi)} |D^\alpha \hat{\phi}(\xi)| < \infty. \tag{2.5}$$

The topology of S_ω is defined by the semi-norms $p_{\alpha,\lambda}$ and $\pi_{\alpha,\lambda}$. The dual of S_ω is denoted by S'_ω the elements of which are called ultra-distributions. We may refer to [1] for its various properties. We note that for $\omega(\xi)$ = log (1+|ξ|), S_ω is reduces to S, the Schwartz space.

We also recall the definition of test function space $\mathbf{D}_\omega$. The space $\mathbf{D}_\omega$ is the set of all ϕ in $L^1(\mathbf{R}^n)$ such that ϕ has compact support and $\|\phi\|_\lambda < \infty$ for all

$\lambda > 0$ and

$$\|\phi\|_\lambda = \int_{R^n} \left|\hat{\phi}(\xi)\right| e^{\lambda\omega(\xi)}\, d\xi \tag{2.6}$$

Now, let $\omega \in M_c$. Then K_ω is defined to be the set of positive function k in $\mathbf{R}^n$ with the following property. There exists $\lambda>0$ such that

$$k(\xi+\eta) \le e^{\lambda\omega(-\xi)} k(\eta)\ \text{ for all }\ \xi,\ \eta \in \mathbf{R}^n. \tag{2.7}$$

Let $\omega \in M_c$, $k \in K_\omega$ and $1 \le p < \infty$. Then weighted Sobolev space $\mathbf{B}^{\omega}_{,k}(\mathbf{R}^n)$ is defined to be the space of all ultra-distributions $f \in S'_\omega$ such that

$$\| f \|_k = \left(\int_{R^n} \left| k(\xi) \hat{f}(\xi) \right|^2 d\xi \right)^{\frac{1}{2}} < \infty; \tag{2.8}$$

and

$$\| f \|_{\infty,k} = \text{ess sup}\ \ k(\xi) \left| \hat{f}(\xi) \right|. \tag{2.9}$$

Note that the space $\mathbf{B}^{\omega}_{k}(\mathbf{R}^n)$ is a generalization of the HÖrmander space $\mathbf{B}_k(\mathbf{R}^n)$ [2] and reduces to the space $B_k(\mathbf{R}^n)$ for ω = log (1+|ξ|).

The Wavelet Transform on Weighted Sobolev Space $\mathbf{B}^{\omega}_{k}$.

In this section we define the space W_k of all measurable functions f on $\mathbf{R}$x$\mathbf{R_0}$ such that

$$\|f(b,a)\|_{W_k} = \left(\int_{R_0} \left(\|f(b,a)\|_k^2 \right) \frac{da}{a} \right)^{\frac{1}{2}} < \infty, \tag{3.1}$$

Theorem3.1. Assume that analyzing wavelet $\psi \in L^2$ satisfies the following admissibility condition:

$$C_\psi = \int_R \frac{\left|\hat{\psi}(\xi)\right|^2}{|\xi|} d\xi < \infty. \tag{3.2}$$

Let $(L_\psi f)(b,a)$ be the wavelet transform of the function $f \in \mathbf{B}_k^\omega$ with respect to the analyzing wavelet $\psi \in L^2$.

Then

$$\left\|\left(L_\psi f\right)(b,a)\right\|_{W_k} = \| f \|_k, \tag{3.3}$$

Proof. From (3.1), we have

$$\left\|\left(L_\psi f\right)(b,a)\right\|_{W_k}^2 = \int_{R_0} \left\|(L_\psi f)(b,a)\right\|_k^2 \frac{da}{a}$$

$$= \int_{R_0} \left(\int_R | k(\xi) |^2 | (L_\psi f)(b,a)^\wedge(\xi) |^2 \ d\xi \frac{da}{a} \right.$$

$$= \int_{R_0} \left(\int_R | k(\xi) |^2 \left(\frac{1}{C_\psi} \right) a \ \ |\hat{\psi}(-a\xi)|^2 | \ \hat{\mathrm{f}}(\xi) |^2 \ d\xi \right) \frac{da}{a}$$

$$= \left(\frac{1}{C_\psi} \right) \int_{R_0} \frac{|\hat{\psi}(-u)|^2}{|u|} du \int_R | k(\xi) \hat{f}(\xi) |^2 \ d\xi$$

$$= \left(\frac{1}{C_\psi} \right) \int_{R_0} \frac{|\hat{\psi}(u)|^2}{|u|} du \ \| f \|_k$$

$$= \left(\frac{1}{C_\psi} \right) C_\varphi \ \| f \|_k = \| \mathrm{f} \|_{\mathrm{k}} \ .$$

Theorem3.2. For admissible and integrable ψ_1, ψ_2 and $\mathrm{f},\mathrm{g} \in \mathbf{B}_k^\omega$.

$$\left\|L_{\psi_1} f(b,a) - L_{\psi_2} g(b,a)\right\|_k \le \left(\left\| \frac{\psi_1}{\sqrt{C_{\psi_1}}} - \frac{\psi_2}{\sqrt{C_{\psi_2}}} \right\|_{L_1} \|f\|_k + \left\| \frac{\psi_2}{\sqrt{C_{\psi_2}}} \right\|_{L_2} \ \| f - g \|_k \right).$$

Proof. We have

$$\left\|L_{\psi_1} f(b,a) - L_{\psi_2} g(b,a)\right\|_k \le \left\|L_{\psi_1} f(b,a) - L_{\psi_2} f(b,a)\right\|_k$$
$$+ \left\|L_{\psi_2} f(b,a) - L_{\psi_2} g(b,a)\right\|_k . \tag{3.4}$$

Now,

$$\| L_{\psi_1} f(b,a) - L_{\psi_2} f(b,a) \|_k = \left(\int_R \left| \left(L_{\psi_1} f(b,a) - L_{\psi_2} f(b,a) \right)^\wedge (\xi) \right|^2 \ | k(\xi) |^2 \ d\xi \right)^{\frac{1}{2}}$$

$$= \left(\int_R \left| \sqrt{\frac{1}{C_{\psi_1}}} \, | a |^{\frac{1}{2}} \, \hat{\psi}_1(-a\xi)\, \hat{f}(\xi) - \sqrt{\frac{1}{C_{\psi_2}}} \, | a |^{\frac{1}{2}} \, \hat{\psi}_2(-a\xi) f(\xi) \right|^2 \, | k(\xi) |^2 \, d\xi \right)^{\frac{1}{2}}$$

$$= \left(\int_R | a | \, |k(\xi)|^2 \, | \hat{f}(\xi) |^2 \, d\xi \left| \left(\frac{\hat{\psi}_1}{\sqrt{C_{\psi_1}}} - \frac{\hat{\psi}_2}{\sqrt{C_{\psi_2}}} \right)(-a\xi) \right|^2 \right)^{\frac{1}{2}}. \qquad (3.5)$$

Now, using the inequality

$$\left| \hat{\psi}(\xi) \right| \leq \| \psi \|_{L^1},$$

we have

$$\left| \frac{\hat{\psi}_1}{\sqrt{C_{\psi_1}}} - \frac{\hat{\psi}_2}{\sqrt{C_{\psi_2}}} \right| \leq \left\| \frac{\psi_1}{\sqrt{C_{\psi_1}}} - \frac{\psi_2}{\sqrt{C_{\psi_2}}} \right\|_{L^1};$$

so that

$$\left| \frac{\hat{\psi}_1}{\sqrt{C_{\psi_1}}} - \frac{\hat{\psi}_2}{\sqrt{C_{\psi_2}}} \right|^2 \leq \left\| \frac{\psi_1}{\sqrt{C_{\psi_1}}} - \frac{\psi_2}{\sqrt{C_{\psi_2}}} \right\|_{L_1}^2. \qquad (3.6)$$

Using (3.6) in (3.5), we have

$$\| L_{\psi_1} f(b,a) - L_{\psi_2} f(b,a) \|_k \leq | a |^{\frac{1}{2}} \left\| \frac{\psi_1}{\sqrt{C_{\psi_1}}} - \frac{\psi_2}{\sqrt{C_{\psi_2}}} \right\|_{L_1} \| f \|_k. \qquad (3.7)$$

Similarly, we can write

$$\| L_{\psi_2} f(b,a) - L_{\psi_2} g(b,a) \|_k \leq \left\| \frac{\psi_2}{\sqrt{C_{\psi_2}}} \right\|_{L_1} | a |^{\frac{1}{2}} \| f - g \|_k. \qquad (3.8)$$

Invoking (3.8), (3.4), we have

$$\| L_{\psi_1} f(b,a) - L_{\psi_2} g(b,a) \|_k \leq | a |^{\frac{1}{2}} \left\| \frac{\psi_1}{\sqrt{C_{\psi_1}}} - \frac{\psi_2}{\sqrt{C_{\psi_2}}} \right\|_{L_1} \| f \|_k.$$

$$+ \left\| \frac{\psi_2}{\sqrt{C_{\psi_2}}} \right\|_{L_1} \| f - g \|_k$$

Asymptotic Behavior for Small Dilation Parameters.

Let us recall the equation (1.9):

$$L_\psi f(b,a)=\frac{1}{\sqrt{C_\psi}}\frac{1}{\sqrt{|a|}}\int_R \psi\left(\frac{t-b}{a}\right)f(t)dt. \qquad (4.1)$$

In what follows we assume that ψ is real valued and a>0.
Let us define

$$\psi_a(x)=\frac{1}{a}\psi\left(\frac{x}{a}\right). \qquad (4.2)$$

Let us use the notation

$$\wedge_\psi f(b,a)=(\psi_a * f)(b)=\frac{1}{a}\int_R \psi\left(\frac{b-t}{a}\right)f(t)dt. \qquad (4.3)$$

From (4.1) and (4.3), we have

$$(\psi_a * f)(b)=(\wedge_\psi f)(b,a)=\sqrt{\frac{C_\psi}{a}}L_\psi f(b,-a). \qquad (4.4)$$

Theorem4.1. Let $f\in \mathbf{B}_k^\omega$ and $\psi\in L^1(R)$ with $\int_R \psi(t)dt=1$.

Then

$$\wedge_\psi f(.,a)\to f(.) \text{ in } \mathbf{B}_k^w \text{ as } a\to 0;$$

Proof. In view of (4.3) we have

(i) $$\|\psi_a * f - f\|_k^2=\int_R \left|(\psi_a * f - f)^\wedge(\xi)\right|^2 |k(\xi)|^2\, d\xi$$

$$=\int_R \left|(\psi_a * f)^\wedge(\xi)-\hat{f}(\xi)\right|^2 |k(\xi)|^2\, d\xi$$

$$=\int_R \left|\left(\frac{C_\psi}{a}\right)^{\frac{1}{2}}((L_\psi f)(b,a))^\wedge(\xi)-\hat{f}(\xi)\right|^2 |k(\xi)|^2 d\xi$$

$$=\int_R \left|\left(\frac{C_\psi}{a}\right)^{\frac{1}{2}}\cdot\left(\frac{1}{C_\psi}\right)^{\frac{1}{2}}|a|^{\frac{1}{2}}\,\hat{\psi}(a\xi)\,\hat{f}(\xi)-\hat{f}(\xi)\right|^2$$

$$=\int_R \left|\hat{\psi}(a\xi)\hat{f}(\xi)-\hat{f}(\xi)\right|^2 |k(\xi)|^2\, d\xi$$

$$=\int_R |k(\xi)|^2\left|\hat{f}(\xi)\right|^2\left|1-\hat{\psi}(a\xi)\right|^2 d\xi$$

$$=\int_R |k(\xi)|^2\,\hat{f}(\xi)\left|1-\hat{\psi}(a\xi)\right|^2 d\xi$$

$$=\int_R |I(a,\xi)|^2\, d\xi,$$

where $|I(a,\xi)| = \left| k(\xi)\hat{f}(\xi)\left(1-\hat{\psi}(a\xi)\right)\right|$.

Then we have $\lim_{a\to 0}|I(a,\xi)| = 0$ a.e.

Let us now set $M = \sup_{\xi\in R}\left|1-\hat{\psi}(a\xi)\right|$, which is independent of a.

Then

$$\left|I(a,\xi)\right| \le M.\left|k(\xi)\hat{f}(\xi)\right|.$$

Now, applying the dominated convergence theorem, we have

$$\left(\psi_a * f\right) = \wedge_\psi f(.,a) \to f(.) \text{ in } \mathbf{B}_k^\omega \text{ as } a \to 0.$$

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